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ALL BOUNDED TYPE SIEGEL DISKS OF RATIONAL MAPS ARE QUASI-DISKS

ALL BOUNDED TYPE SIEGEL DISKS OF RATIONAL MAPS ARE QUASI-DISKS

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Abstract. We prove that every bounded type Siegel disk of a rational map must be a quasi-disk with at least one critical point on its boundary. This verifies Douady-Sullivan's conjecture in the case of bounded type rotation numbers.

1. Introduction

A Siegel disk of a rational map f is a maximal domain on which f is holomorphically conjugate to an irrational rotation. It was conjectured by Douady and Sullivan in 1980's that the boundary of every Siegel disk for a rational map has to be a Jordan curve [6]. This has remained an open problem, even for quadratic polynomials. The main purpose of this paper is to verify this conjecture under the condition that the rotation number of the Siegel disk is of bounded type. Here we say an irrational number $0 < \theta < 1$ is of bounded type if $\sup\{a_k\} < \infty$ where $\theta = [a_1, \cdots, a_n, \cdots,]$ is the continued fraction of θ. Before we state the main result of the paper, let us give a brief account of the previous studies on this problem.

In 1986, Douady observed that quasisymmetric linearization of critical circle mappings would imply that the boundary of the Siegel disk of a quadratic polynomial is a quasi-circle. Using work of Swiatek, Herman then proved the required quasisymmetric linearization result for analytic circle mappings with bounded type rotation numbers. This implies that every bounded type Siegel disk of a quadratic polynomial must be a quasi-disk whose boundary passes through the unique finite critical point of the quadratic polynomial [7]. In 1998, by considering a surgery map defined on certain space of some degree-5 Blaschke products, Zakeri extended Douady-Herman's result to bounded type Siegel disks of all cubic polynomials [17]. Shortly after that, in his webpage Shishikura announced

Theorem (Shishikura). *All bounded type Siegel disks of polynomial maps are quasi-disks which have at least one critical point on their boundaries.*

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The main purpose of this paper is to generalize the above result to bounded type Siegel disks of all rational maps.

Main Theorem. *Let $d \geq 2$ be an integer and $0 < \theta < 1$ be an irrational number of bounded type. Then there exists a constant $1 < K(d, \theta) < \infty$ depending only on d and θ such that for any rational map f of degree d, if f has a fixed Siegel disk with rotation number θ, then the boundary of the Siegel disk is a $K(d, \theta)$-quasi-circle which passes through at least one critical point of f.*

There are two main ingredients in the proof of the Main Theorem. The first one is due to Shishikura by which he proved that bounded type Siegel disks of polynomial maps are all quasi-disks. The idea of Shishikura is to prove that any invariant curve inside a bounded type Siegel disk of a polynomial map is uniformly quasiconformal. The result then follows by letting the invariant curve approach the boundary of the Siegel disk. A detailed description of this strategy will be given in §3 of this paper.

The second one is an extension of Herman's uniform quasisymmetric bound to all analytic circle mappings induced by *centered* Blaschke products (for the definition of *centered* Blaschke products, see §2). As indicated by Shishikura, the key tool used in his proof is a uniform quasisymmetric bound of the linearization maps for a compact family of analytic circle mappings, which was due to Herman (see Theorem A of §2). The main obstruction in generalizing Shishikura's result to all rational maps is that the family of Blaschke products involved in constructing Siegel disks of rational maps is not compact anymore, and Herman's theorem does not apply directly in this situation. The core of our proof is an extension of Herman's theorem to all centered Blaschke products (see Theorem B of §2). This is the heart of the whole paper. One of the key tools used in our proof is the Relative Schwarz Lemma proved by Buff and Chéritat in [2].

The following is a sketch of the organization of the paper.

In §2, we introduce Herman's theorem and its extension (Theorem A and Theorem B). Since the proof of Theorem B is quite long, we postpone it until the last section of the paper.

In §3, we prove the Main Theorem. The proof is divided into two steps. In the first step, we prove the Main Theorem under the condition that the post-critical set of the rational map does not intersect the interior of the Siegel disk (Lemma 3.6). In the second step we prove the Main Theorem in the general case (Lemma 3.8). The proof of Lemma 3.6 is based on Theorem B and Shishikura's strategy. The proof of Lemma 3.8 uses Lemma 3.6 and a trick of holomorphic motion.

In §4, we prove Theorem B and thus complete the proof of the Main Theorem.

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2. Herman's Theorem and its extension

Let $m = 2d - 1$ with $d \geq 2$ being some integer. Let $\theta = [a_1, \cdots, a_n, \cdots]$ be an irrational number with $\sup\{a_n\} < \infty$. We call such θ of bounded type. Let $\mathbb{T}$ denote the unit circle and $R_\theta : \mathbb{T} \to \mathbb{T}$ denote the rigid rotation given by $z \to e^{2\pi i\theta} z$. Let $\mathbf{H}_\theta^m$ denote the class of all the Blaschke products

$$B(z) = \lambda z^d \prod_{i=1}^{d-1} \frac{1 - \overline{a}_i z}{z - a_i}, \tag{1}$$

such that

1. $|a_i| < 1$ for all $1 \leq i \leq d - 1$,
2. $|\lambda| = 1$,
3. $B|\mathbb{T} : \mathbb{T} \to \mathbb{T}$ is a circle homeomorphism of rotation number θ.

In one of his three handwritten manuscripts [9](see also [3] and [4]), Herman proved

Theorem A. *Let $m \geq 3$ be an odd integer and $0 < \theta < 1$ be an irrational number of bounded type. Then there is a constant $1 < K(m,\theta) < \infty$ depending only on m and θ such that for any $B \in \mathbf{H}_\theta^m$, there is a $K(m,\theta)$-quasi-symmetric homeomorphism h_B of the unit circle such that $B|\mathbb{T} = h_B^{-1} \circ R_\theta \circ h_B$ and $h_B(1) = 1$, where $R_\theta : z \mapsto e^{2\pi i\theta} z$ is the rigid rotation given by θ.*

The proof of Theorem A in [9] depends essentially on the fact that the family $\mathbf{H}_\theta^d$ is compact in the following sense.

Lemma 2.1. *There is an annular neighborhood H of $\mathbb{T}$, such that*

1. *all maps in $\mathbf{H}_\theta^m$ are holomorphic in H, and*
2. *for any sequence $\{B_n\} \subset \mathbf{H}_\theta^m$, there is a subsequence $\{B_{n_k}\}$ such that $B_{n_k}|H$ converges uniformly to $B|H$ where $B \in \mathbf{H}_\theta^l$ and $1 \leq l \leq m$ is some odd integer.*

Proof. By §15 of [9], there is a $0 < \rho < 1$ such that for any $B \in \mathbf{H}_\theta^m$ given by (1), one has $|a_i| \leq \rho$. Let

$$H = \{z \mid (1 + \rho)/2 < |z| < 2\}.$$

Then all the maps in $\mathbf{H}_\theta^m$ are holomorphic in H. This proves the first assertion. Let

$$B_n(z) = \lambda z^d \prod_{i=1}^{d-1} \frac{1-\overline{a}_{n,i} z}{z - a_{n,i}}.$$

Since $|a_{n,i}| \le \rho$, there is a subsequence of integers $\{n_k\}$ such that for each $1 \le i \le d-1$, $a_{n_k,i} \to b_i$ with $0 \le |b_i| \le \rho$. It follows that as $k \to \infty$,

$$\frac{1-\overline{a}_{n_k,i} z}{z - a_{n_k,i}} \to \frac{1-\overline{b}_i z}{z - b_i}$$

uniformly on H. Let

$$B(z) = \lambda z^d \prod_{i=1}^{d-1} \frac{1-\overline{b}_i z}{z - b_i}.$$

Then $B \in \mathbf{H}_\theta^l$ with $1 \le l \le m$ being some odd integer and $B_{n_k} \to B$ uniformly on H. This proves the second assertion and Lemma 2.1 follows.

□

Theorem A plays an important role in the study of bounded type Siegel disks of polynomial maps. Among all of those the most remarkable one is Shishikura's result which says that any bounded type Siegel disk of a polynomial map is a quasi-circle with at least one critical point on it.

Let d, m and θ be as above. Let $\mathbf{B}_\theta^m$ denote the class of all the Blaschke products

$$B(z) = \lambda \prod_{i=1}^{d} \frac{z-p_i}{1-\overline{p}_i z} \prod_{j=1}^{d-1} \frac{z-q_j}{1-\overline{q}_j z} \tag{2}$$

such that

1. $|p_i| < 1$ and $|q_j| > 1$ for all $1 \le i \le d$ and $1 \le j \le d-1$,
2. $|\lambda| = 1$,
3. $B|\mathbb{T} : \mathbb{T} \to \mathbb{T}$ is a circle homeomorphism of rotation number θ.

For any $B \in \mathbf{B}_\theta^m$, by Herman's result in [9] it is known that the analytic circle mapping

$$B|\mathbb{T} : \mathbb{T} \to \mathbb{T}$$

is quasisymmetrically conjugate to the rigid rotation $R_\theta : z \mapsto e^{2\pi i \theta}$. Then $B|\mathbb{T}$ has a unique invariant probability measure on $\mathbb{T}$ which has no atoms. Let us denote it by μ_B. According to Douady and Earle [8], to such μ_B, one can assign a vector field ξ_{μ_B} on Δ as follows,

$$\xi_{\mu_B}(z) = (1-|z|^2) \int_{\mathbb{T}} \frac{\zeta - z}{1-\bar{z}\zeta} d\mu_B(\zeta), \;\; z \in \Delta.$$

By Proposition 1 of [8], the vector field ξ_{μ_B} has a unique zero in Δ, which is called the *conformal barycenter* of μ_B. Let us denote it by z_B. From the above formula it follows that $z_B = 0$ if and only if

$$\int_{\mathbb{T}} \zeta \, d\mu_B(\zeta) = 0. \tag{3}$$

Note that for any Möbius map g which maps the unit circle to itself and preserves the orientation, $g_*\mu_B$ is the unique invariant probability measure for the analytic circle mapping $(g \circ B \circ g^{-1})|\mathbb{T} : \mathbb{T} \to \mathbb{T}$. It is clear that $g_*\mu_B$ has no atoms. According to [8], the assignment of $\mu \mapsto \xi_\mu$ is conformally natural in the following sense: if g is a Möbius map which maps the unit circle to itself and preserves the orientation, then

$$\xi_{g^*_{\mu_B}}(z) = g'(g^{-1}(z)) \cdot \xi_{\mu_B}(g^{-1}(z)).$$

It follows that if g maps z_B to 0, then the conformal barycenter of $g_*\mu_B$ is 0.

Definition 2.1. We say B is a centered Blaschke product if $z_B = 0$.

From the previous observation, any Blaschke product in $\mathbf{B}_\theta^m$ is conjugate to a centered Blaschke product by a Möbius map which maps the unit circle to itself and preserves the orientation. The core of the proof of our Main Theorem is the extension of Herman's theorem to all the centered Blaschke products in $\mathbf{B}_\theta^m$.

Theorem B. *Let $m \geq 3$ be an odd integer and $\theta = [a_1, \cdots, a_n, \cdots]$ be a bounded type irrational number. Then there is a constant $1 < M(m, \theta) < \infty$ depending only on m and θ such that for any centered Blaschke product B in $\mathbf{B}_\theta^m$, the map*

$$h_B : \mathbb{T} \to \mathbb{T}$$

is an $M(m, \theta)$-quasisymmetric homeomorphism, where $h_B : \mathbb{T} \to \mathbb{T}$ is the circle homeomorphism such that $B|\mathbb{T} = h_B^{-1} \circ R_\theta \circ h_B$ and $h_B(1) = 1$.

Remark 1. We would like to remark that for every odd integer $m \geq 3$ and irrational rotation number $0 < \theta < 1$, the family of centered Blaschke products in $\mathbf{B}_\theta^m$ is not compact in the sense of Lemma 2.1. One can show that for any annular neighborhood H of the unit circle, there is a centered Blaschke product B in $\mathbf{B}_\theta^m$ such that B is not holomorphic in H.

As an immediate corollary of Theorem B, we have

Corollary 2.1. Let $m = 2d-1 \geq 3$ be an odd integer and $\theta = [a_1, \cdots, a_n, \cdots]$ be a bounded type irrational number. Then there is a constant $1 < K(d, \theta) < \infty$ depending only on d and θ such that for any Blaschke product B in $\mathbf{B}_\theta^m$, the map

$$h_B : \mathbb{T} \to \mathbb{T}$$

can be extended to a $K(d,\theta)$-quasiconformal homeomorphism of the unit disk to itself, where $h_B : \mathbb{T} \to \mathbb{T}$ is the circle homeomorphism such that $B|\mathbb{T} = h_B^{-1} \circ R_\theta \circ h_B$ and $h_B(1) = 1$.

3. Proof of The Main Theorem assuming Theorem B

Let $d \ge 2$ and $0 < \theta < 1$ be an irrational number of bounded type. Suppose that f is a rational map of degree d and has a fixed Siegel disk D centered at the origin and with rotation number θ. By a Möbius conjugation, we may assume that $\overline{D}$ is contained in a compact set of the complex plane. Let Δ denote the unit disk. Let

$$\lambda : \Delta \to D$$

be the holomorphic isomorphism such that $\lambda(0) = 0$, $\lambda'(0) > 0$, and

$$\lambda^{-1} \circ f \circ \lambda(z) = e^{2\pi i\theta} z$$

for all $z \in \Delta$. For $0 < r < 1$, let

$$\Gamma_r = \{\lambda(re^{it}) \bigm| 0 \le t \le 2\pi\}.$$

Let $K > 1$ and $\widehat{\mathbb{C}}$ be the Riemann sphere. We call a simple closed curve $\Gamma \subset \widehat{\mathbb{C}}$ a K-quasi-circle if there is a K-quasiconformal homeomorphism

$$\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

such that $\Gamma = \phi(\mathbb{T})$ where $\mathbb{T}$ is the unit circle.

Lemma 3.1. *If there exists a $1 < K < \infty$ such that Γ_r is a K-quasi-circle for all $0 < r < 1$, then ∂D is a K-quasi-circle. In particular, the map $f|\partial D : \partial D \to \partial D$ is injective, and thus ∂D contains at least one of the critical points of f.*

Proof. By assumption, for any integer $n > 1$, there is a K-quasiconformal homeomorphism $\sigma_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$\sigma_n(\mathbb{T}) = \Gamma_{1-1/n}.$$

We may assume that σ_n maps the origin into the inside of $\Gamma_{1-1/n}$. Let η_n be a Möbius map which preserves the unit disk and maps the origin to $\sigma_n^{-1}(0)$. Let

$$\omega_n = \sigma_n \circ \eta_n.$$

Then ω_n is a K-quasiconformal homeomorphism of the sphere and moreover, $\omega_n(\mathbb{T}) = \Gamma_{1-1/n}$ and $\omega_n(0) = 0$. It follows that any limit map of the sequence $\{\omega_n\}$ is a K-quasiconformal homeomorphism of the sphere. By taking a convergent subsequence, we may assume that there is a K-quasi-conformal homeomorphism

$$\omega : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

such that ω_n converges uniformly to ω with respect to the spherical metric.

We claim that

$$D = \omega(\Delta).$$

Let us prove the claim now. For $r > 0$, let Δ_r denote the Euclidean disk centered at the origin and with radius r. Then for any $1 < n < l$ we have $\lambda(\Delta_{1-1/n}) \subset \lambda(\Delta_{1-1/l})$. Since $\omega_l(\Delta) = \lambda(\Delta_{1-1/l})$, we have

$$\lambda(\Delta_{1-1/n}) \subset \omega_l(\Delta). \tag{4}$$

Let us first prove that

$$\lambda(\Delta_{1-1/n}) \subset \omega(\Delta). \tag{5}$$

Suppose (5) were not true. Since $\lambda(\Delta_{1-1/n})$ is open and $\omega(\Delta)$ is a quasi-disk, there would be a point $z \in \lambda(\Delta_{1-1/n})$ such that $d(z, \overline{\omega(\Delta)}) = \delta > 0$. Here $d(\cdot,\cdot)$ denotes the distance with respect to the spherical metric. Since $\omega_l \to \omega$ uniformly with respect to the spherical metric, we have

$$d(z, \overline{\omega_l(\Delta)}) > \delta/2 > 0$$

for all l large enough. This is a contradiction with (4). Thus (5) has been proved. Since $D = \lambda(\Delta)$, by letting $n \to \infty$ in the left hand of (5), we get

$$D \subset \omega(\Delta). \tag{6}$$

Note that for any $l \geq 1$, we have

$$\omega_l(\Delta) = \lambda(\Delta_{1-1/l}) \subset \lambda(\Delta) = D. \tag{7}$$

For any $z \in \Delta$, let $H = \{\zeta \mid |z| < |\zeta| < 1\}$. Since $\omega_l(0) = 0$, $\omega_l(H)$ is an annulus contained in D which separates $\{0, \omega_l(z)\}$ and ∂D. Since ω_l is K-quasiconformal for all l, it follows that

$$\mathrm{mod}(\omega_l(H)) \geq \frac{1}{K}\mathrm{mod}(H) = \frac{1}{2K\pi}\log\frac{1}{|z|}.$$

This implies that there is some $\delta > 0$ independent of l such that

$$d(\omega_l(z), \partial D) \geq \delta$$

for all l. Since $\omega_l(z) \in D$, it follows that $B_\delta(w_l(z)) \subset D$ for all l. Since $\omega_l \to \omega$ uniformly with respect to the spherical metric, it follows that $w(z) \in D$. Since z is arbitrary, we have

$$\omega(\Delta) \subset D. \tag{8}$$

From (6) and (8) it follows that $D = \omega(\Delta)$ and the claim has been proved.

From the claim we have $\partial D = \omega(\mathbb{T})$. Since ω is a K-quasiconformal homeomorphism of the sphere to itself, it follows that ∂D is a K-quasi-circle and D is a K-quasi-disk. Since $\lambda : \Delta \to D$ is a holomorphic isomorphism, one can homeomorphically extended λ to $\partial\Delta$. So we have

$$\lambda^{-1} \circ f \circ \lambda(z) = e^{2\pi i\theta} z$$

holds for all $z \in \partial\Delta$. This implies that

$$f|\partial D : \partial D \to \partial D$$

is injective. By a result of Herman (see [10]), it follows that ∂D contains at least one of the critical points of f. This completes the proof of Lemma 3.1. $\square$

Let $0 < r < 1$ and let

$$D_r = \{\lambda(se^{it}) \bigm| 0 \le s < r,\ 0 \le t \le 2\pi\}.$$

Let

$$\phi : \widehat{\mathbb{C}} \setminus \overline{\Delta} \to \widehat{\mathbb{C}} \setminus \overline{D_r}$$

be the holomorphic isomorphism such that $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$. Take $r < R < 1$. Let

$$\Theta_R = \phi^{-1}(\Gamma_R).$$

Then Θ_R is a real-analytic simple closed curve which surrounds the closed unit disk. Let Θ_R^* denote the symmetric image of Θ_R about the unit circle. Let Λ_R denote the bounded component of $\widehat{\mathbb{C}} \setminus \Theta_R^*$. It is clear that Λ_R is a Jordan domain with smooth boundary which lies in the inside of the unit disk and contains the origin. Let

$$A_R = \Delta \setminus \overline{\Lambda_R}$$

be the annulus bounded by $\mathbb{T}$ and Θ_R^*.

Take $r_0 > 0$ small enough such that $\overline{\Delta_{r_0}} \subset D_r$ where $\Delta_{r_0} = \{z \mid |z| < r_0\}$. Let $\eta : \Lambda_R \to \Delta_{r_0}$ be the Riemann isomorphism such that $\eta(0) = 0$ and $\eta'(0) > 0$. Since $\partial\Lambda_R$, $\partial\Delta_{r_0}$, $\partial\Delta$ and ∂D_r are all smooth curves, there is a quasiconformal homeomorphism $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

1. $\Phi(z) = \phi(z)$ in the outside of the unit disk, and
2. $\Phi(z) = \eta(z)$ in Λ_R, and
3. Φ is quasiconformal in A_R.

For $\zeta \in \mathbb{C} \cup \{\infty\}$, let $\zeta^* = 1/\bar{\zeta}$ be the symmetric image of ζ about the unit circle. Define

$$G(z) = \begin{cases} \Phi^{-1} \circ f \circ \Phi(z) & \text{for } |z| \ge 1, \\ (\Phi^{-1} \circ f \circ \Phi(z^*))^* & \text{for } |z| < 1\ . \end{cases} \tag{9}$$

Let

$$H_r = \Phi^{-1}(\{\lambda(se^{it}) \bigm| r \le s < 1, 0 \le t \le 2\pi\}.$$

Let H_r^* denote the symmetric image of H_r about the unit circle. Then $H_r \cup H_r^*$ is an annular neighborhood of the unit circle. Throughout the following, let us set

$$m = 2d - 1.$$

By the construction, we have

Lemma 3.2. *The map G is a degree m branched covering map of the sphere to itself which is holomorphic in $H_r \cup H_r^*$. Moreover, G is holomorphically conjugate to the rigid rotation $z \mapsto e^{2\pi i\theta}z$ in $H_r \cup H_r^*$.*

From the construction we see if G is quasiconformal at some point z, then $G(z)$ lies in $H_r \cup H_r^*$. Let μ_0 denote the standard complex structure in $H_r \cup H_r^*$. Let $G_0 = G|(H_r \cup H_r^*)$ denote the restriction of G to $H_r \cup H_r^*$. By Lemma 3.2 the map

$$G_0 : H_r \cup H_r^* \to H_r \cup H_r^*$$

is a holomorphic isomorphism and μ_0 is G_0-invariant. So one can pull back μ_0 by the iteration of G to get a G-invariant complex structure μ on the whole sphere $\widehat{\mathbb{C}}$. It follows from the symmetric property of G and μ_0 that μ is symmetric about the unit circle.

Note that if G is quasiconformal at some point z with $|z| > 1$, then $G(z)$ actually belongs to H_r^* which is contained in the inside of the unit disk. This implies

Lemma 3.3. *For almost every z in the outside of the unit disk, if $\mu(z) \neq 0$, then there exists some integer $k \geq 1$ such that $G^k(z) \in \Delta$.*

Let Ψ denote the quasiconformal homeomorphism which solves the Beltrami equation given by μ and fixes 0, 1, and the infinity. Let

$$B(z) = \Psi \circ G \circ \Psi^{-1}(z).$$

Since μ is symmetric about the unit circle, the map

$$z \mapsto (\Psi(z^*))^*$$

is also a quasiconformal homeomorphism of the sphere to itself which has Beltrami coefficient μ. Note that it also fixes 0, 1 and the infinity. So $\Psi(z) = (\Psi(z^*))^*$ for all $z \in \widehat{\mathbb{C}}$. Since $G(z^*) = (G(z))^*$ for all $z \in \widehat{\mathbb{C}}$, it follows that $B(z^*) = (B(z))^*$ for all $z \in \widehat{\mathbb{C}}$. This implies that

Lemma 3.4. *$B \in \mathbf{B}_\theta^m$.*

Lemma 3.5. *For almost every z in the outside of the unit disk, if Ψ^{-1} is not conformal at z then there is some integer $k \geq 1$ such that $B^k(z) \in \Delta$.*

Proof. Note that Ψ^{-1} is not conformal at z for some $|z| > 1$ if and only if Ψ is not conformal at $\Psi^{-1}(z)$. From Lemma 3.3 it follows that there is some integer $k \geq 1$ such that $G^k(\Psi^{-1}(z)) \in \Delta$. By the symmetric property of Ψ, Ψ preserves the unit circle and thus maps the unit disk homeomorphically onto the unit disk. We thus have

$$B^k(z) = (\Psi \circ G^k \circ \Psi^{-1})(z) \in \Delta.$$

The lemma follows. □

Let $h_B : \mathbb{T} \to \mathbb{T}$ be the circle homeomorphism such that $h_B(1) = 1$ and

$$B|\mathbb{T} = h_B^{-1} \circ R_\theta \circ h_B.$$

By Corollary 2.1, one can extend h_B to a $K(d, \theta)$-quasiconformal homeomorphism

$$H_B : \Delta \to \Delta$$

where $1 < K(d, \theta) < \infty$ is some constant depending only on d and θ. Now let us define the modified Blaschke product as follows.

$$\widehat{B}(z) = \begin{cases} B(z) & \text{for } |z| \geq 1, \\ H_B^{-1} \circ R_\theta \circ H_B(z) & \text{for } z \in \Delta. \end{cases}$$

From the above construction, we have

Proposition 3.1. Let $z \in \widehat{\mathbb{C}} \setminus \Delta$. Then

$$\widehat{B}(z) \notin \Delta \Leftrightarrow f(\Phi \circ \Psi^{-1}(z)) \notin D_r.$$

Moreover, if $\widehat{B}(z) \notin \Delta$, then

$$\Phi \circ \Psi^{-1}(\widehat{B}(z)) = f(\Phi \circ \Psi^{-1}(z)).$$

Let $\Omega_{\widehat{B}}$ and Ω_f denote critical sets of $\widehat{B}$ and f, respectively. Let

$$P_{\widehat{B}} = \bigcup_{k \geq 1}^{\infty} \Omega_{\widehat{B}} \text{ and } P_f = \bigcup_{k \geq 1}^{\infty} \Omega_f$$

denote the post-critical sets of $\widehat{B}$ and f, respectively.

Lemma 3.6. *There is a constant* $1 < K(d, \theta) < \infty$ *depending only on* d *and* θ *such that for any* $0 < r < 1$*, if* $P_f \cap D_r = \emptyset$*, then* Γ_r *is a* $K(d, \theta)$*-quasi-circle.*

Proof. It suffices to prove the lemma under the stronger assumption that $P_f \cap \overline{D_r} = \emptyset$. This is because $P_f \cap \overline{D_{r'}} \subset P_f \cap D_r = \emptyset$ for all $0 < r' < r$, and by the same reasoning as in the proof of Lemma 3.1, one can show that Γ_r must be a $K(d, \theta)$-quasi-circle if $\Gamma_{r'}$ is a $K(d, \theta)$-quasi-circle for all $0 < r' < r$.

Let $\lambda_r : \Delta \to D_r$ be the holomorphic isomorphism such that

$$\lambda_r(1) = \Phi \circ \Psi^{-1}(1)$$

and

$$\lambda_r^{-1} \circ f \circ \lambda_r(z) = e^{2\pi i \theta} z$$

for all $z \in \Delta$. Define a quasiconformal homeomorphism $\chi_0 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$\chi_0(z) = \begin{cases} \Phi \circ \Psi^{-1}(z) & \text{for } |z| \geq 1, \\ \lambda_r \circ H_B(z) & \text{for } z \in \Delta. \end{cases}$$

Let μ_0 denote the complex dilatation of χ_0 and let

$$M = \frac{1 + \|\mu_0\|_\infty}{1 + \|\mu_0\|_\infty}.$$

Then χ_0 is an M-quasiconformal homeomorphism of the sphere which maps the unit disk homeomorphically onto D_r. Note that M depends on r and may go to infinity as $r \to 1$.

Now for every $k \geq 1$, we will define an M-quasiconformal homeomorphism $\chi_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as follows. Note that $P_f \cap \overline{D_r} = \emptyset$ by the assumption in the beginning of the proof. Since $\Phi \circ \Psi^{-1}$ is a bijection between $\Omega_{\widehat{B}}$ and Ω_f, from Proposition 3.1 it follows that $P_{\widehat{B}} \cap \overline{\Delta} = \emptyset$ and thus

$$\Phi \circ \Psi^{-1}(P_{\widehat{B}}) = P_f.$$

So for every $k \geq 1$, if an inverse branch of $\widehat{B}^k$ maps Δ to some domain in the outside of the unit disk, then this inverse branch is univalently defined in an open neighborhood of the closed unit disk. This implies that each component of $\widehat{B}^{-k}(\Delta)$ is a Jordan domain with boundary being real analytic, and moreover, the closures of these Jordan domains are disjoint with each other.

Suppose $\widehat{B}^{-k}(\Delta)$ has l_k components with $l_k \geq 1$ being some integer. Let

$$U_i, \ 1 \leq i \leq l_k$$

denote all the components of $\widehat{B}^{-k}(\Delta)$. By the construction of $\widehat{B}$, it follows that

$$\Phi \circ \Psi^{-1}(U_i), \ 1 \leq i \leq l_k$$

are all the components of $f^{-k}(D_r)$. Let us first define χ_k on each U_i.

If $U_i = \Delta$, define

$$\chi_k|\Delta = \lambda_r \circ H_B.$$

Otherwise, there is a least integer $1 \leq k_0 \leq k$ such that $\widehat{B}^{k_0}(U_i) = \Delta$. Since $P_{\widehat{B}} \cap \overline{\Delta} = P_f \cap \overline{D_r} = \emptyset$, the two maps

$$\widehat{B}^{k_0} : U_i \to \Delta$$

and

$$f^{k_0} : \Phi \circ \Psi^{-1}(U_i) \to D_r$$

are both holomorphic isomorphisms. So one can lift the quasiconformal homeomorphism $\lambda_r \circ H_B : \Delta \to D_r$ to a quasiconformal homeomorphism $\tau_i : U_i \to \Phi \circ \Psi^{-1}(U_i)$ such that the following diagram commutes.

$$\begin{tikzcd}
U_i \arrow[r, "\tau_i"] \arrow[d, "\widehat{B}^{k_0}"'] & \Phi \circ \Psi^{-1}(U_i) \arrow[d, "f^{k_0}"] \\
\Delta \arrow[r, "\lambda_r \circ H_B"] & D_r
\end{tikzcd}$$

In particular, the dilatation of τ_i on U_i is equal to that of $\lambda_r \circ H_B$ on Δ.

Since both ∂U_i and $\Phi \circ \Psi^{-1}(\partial U_i)$ are quasi-circles (in fact, both of them are real analytic curves), τ_i can be homeomorphically extended to ∂U_i. Note that $\widehat{B}^{k_0}(\partial U_i) = \partial\Delta \subset \widehat{\mathbb{C}} \setminus \Delta$ and thus $\widehat{B}^{k}(\partial U_i) \subset \widehat{\mathbb{C}} \setminus \Delta$ for all $k \ge 0$, by Proposition 3.1, the following diagram commutes.

$$\begin{tikzcd}
\partial U_i \arrow[r, "\Phi\circ\Psi^{-1}"] \arrow[d, "\widehat{B}^{k_0}"'] & \Phi\circ\Psi^{-1}(\partial U_i) \arrow[d, "f^{k_0}"] \\
\partial\Delta \arrow[r, "\Phi\circ\Psi^{-1}"] & \partial D_r
\end{tikzcd}$$

Since $\Phi\circ\Psi^{-1}|\partial\Delta = \lambda_r \circ H_B|\partial\Delta$, from the above two diagrams it follows that $\tau_i|\partial U_i = \Phi\circ\Psi^{-1}|\partial U_i$. For each such U_i, define $\chi_k = \tau_i$ on U_i.

Finally let us define $\chi_k = \Phi\circ\Psi^{-1}$ on the complement of $\widehat{B}^{-k}(\Delta)$. Since all ∂U_i, $1 \le i \le l_k$, are quasi-circles which are disjoint with each other, χ_k is a quasiconformal homeomorphism of the sphere to itself. In this way we get a sequence of quasiconformal homeomorphisms $\chi_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, $k \ge 0$. We claim

1. $\chi_k(\Delta) = D_r$,
2. χ_k is an M-quasiconformal homeomorphism of the sphere to itself, and
3. the following diagram commutes.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \arrow[r, "\chi_{k+1}"] \arrow[d, "\widehat{B}"'] & \widehat{\mathbb{C}} \arrow[d, "f"] \\
\widehat{\mathbb{C}} \arrow[r, "\chi_k"] & \widehat{\mathbb{C}}
\end{tikzcd} \tag{10}$$

Let us prove the claim now. The first assertion is obvious since by the construction of χ_k, $\chi_k|\Delta = \lambda_r \circ H_B$ for all $k \ge 0$. Again by the construction of χ_k, the dilatation of χ_k on $\widehat{B}^{-k}(\Delta)$ is not greater than the dilatation of $\lambda_r \circ H_B$ on Δ, and the dilatation of χ_k on $\widehat{\mathbb{C}} \setminus \widehat{B}^{-k}(\Delta)$ is not greater than the dilatation of $\Phi\circ\Psi^{-1}$ on $\widehat{\mathbb{C}}\setminus\widehat{B}^{-k}(\Delta)$. So for every $k \ge 1$, the dilatation of χ_k is not greater than the dilatation of χ_0 which is M-quasiconformal. The second assertion then follows. By the construction of χ_k and χ_{k+1}, the following diagram commutes.

$$\begin{tikzcd}
\widehat{B}^{-(k+1)}(\Delta) \arrow[r, "\chi_{k+1}"] \arrow[d, "\widehat{B}"'] & f^{-(k+1)}(D_r) \arrow[d, "f"] \\
\widehat{B}^{-k}(\Delta) \arrow[r, "\chi_k"] & f^{-k}(D_r)
\end{tikzcd}$$

By Proposition 3.1 the following diagram commutes.

$$\begin{tikzcd}
\widehat{\mathbb{C}} \setminus \widehat{B}^{-(k+1)}(\Delta) \arrow[r, "\Phi\circ\Psi^{-1}"] \arrow[d, "\widehat{B}"'] & \widehat{\mathbb{C}} \setminus f^{-(k+1)}(D_r) \arrow[d, "f"] \\
\widehat{\mathbb{C}} \setminus \widehat{B}^{-k}(\Delta) \arrow[r, "\Phi\circ\Psi^{-1}"] & \widehat{\mathbb{C}} \setminus f^{-k}(D_r)
\end{tikzcd}$$

But on $\widehat{\mathbb{C}} \setminus \widehat{B}^{-(k+1)}(\Delta)$, $\chi_{k+1} = \Phi \circ \Psi^{-1}$ and on $\widehat{\mathbb{C}} \setminus \widehat{B}^{-k}(\Delta)$, $\chi_k = \Phi \circ \Psi^{-1}$. This, together with the above two diagrams, implies the third assertion. The claim has been proved.

Now for $k \geq 0$, let μ_k denote the Beltrami coefficient of χ_k. It follows that

$$\|\mu_k\|_\infty \leq \frac{M-1}{M+1} \tag{11}$$

holds for all $k \geq 0$.

Now let ν denote the complex dilatation of $\lambda_r \circ H_B$ which is defined in the inside of the unit disk. Since λ_r is conformal in $\Delta = H_B(\Delta)$, it follows that ν is equal to the complex dilatation of H_B. So ν is $\widehat{B}$-invariant. Since H_B is $K(d, \theta)$-quasiconformal, we have

$$\|\nu\|_\infty \leq \frac{K(d,\theta) - 1}{K(d,\theta) + 1}.$$

Now let $\Sigma \subset \widehat{\mathbb{C}} \setminus \Delta$ be the set consisting of all the points z such that $\widehat{B}^k(z) \notin \Delta$ for all $k \geq 1$. By Lemma 3.5, it follows that for almost every $z \in \Sigma$, Ψ^{-1} is conformal at z. Since $\Psi^{-1}(\widehat{\mathbb{C}} \setminus \Delta) = \widehat{\mathbb{C}} \setminus \Delta$, and since Φ is conformal in the outside of the unit disk, it follows that $\Phi \circ \Psi^{-1}$ is conformal at almost every $z \in \Sigma$. Now let

$$\Xi = \bigcup_{l=0}^{\infty} \widehat{B}^{-l}(\partial \Delta).$$

Then Ξ is the union of countably many real analytic curves and thus is a zero measure set. It is easy to see that for every $z \in \Sigma \setminus \Xi$ and every $k \geq 0$, there is an open neighborhood of such z, say $B_z(r)$, such that $B_z(r) \cap \widehat{B}^{-k}(\Delta) = \emptyset$. By the construction of χ_k, it follows that

$$\chi_k | B_z(r) = \Phi \circ \Psi^{-1} | B_z(r).$$

This implies that the complex dilatation of χ_k is equal to that of $\Phi \circ \Psi^{-1}$ at z. In particular, this implies that for almost every $z \in \Sigma$, $\mu_k(z) = 0$ for all $k \geq 0$.

Now suppose $z \in \Sigma$. Then there is some integer $N \geq 1$ such that

$$\widehat{B}^N(z) \in \Delta.$$

By the construction of the maps $\{\chi_k\}$, $\mu_N(z)$ is the pull back of $\nu(\widehat{B}^N(z))$ by $\widehat{B}^N$, and $\mu_k(z) = \mu_N(z)$ for all $k > N$. Since $\widehat{B}$ is holomorphic in the outside of the unit disk, we thus have for all $k \geq N$,

$$|\mu_k(z)| = |\mu_N(z)| = |\nu(\widehat{B}^N(z))| \leq \frac{K(d,\theta) - 1}{K(d,\theta) + 1}.$$

Now let us define a Beltrami coefficient $\mu(z)$ on the whole Riemann sphere by setting

$$\mu(z) = 0$$

if $z \in \Sigma$ and

$$\mu(z) = \mu_N(z)$$

if $\widehat{B}^N(z) \in \Delta$ for some $N \geq 0$. It follows that

$$\|\mu\|_\infty \leq \frac{K(d,\theta)-1}{K(d,\theta)+1}$$

and $\mu_k(z) \to \mu(z)$ for almost every $z \in \widehat{\mathbb{C}}$. Now from (11) and the fact that $\chi_k|\Delta = \lambda_r \circ H_B$ for all $k \geq 0$, it follows that there is a $K(d,\theta)$-quasiconformal homeomorphism $\chi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that χ_k converges uniformly to χ with respect to the spherical metric. In particular, we have

$$\chi|\Delta = \chi_k|\Delta = \lambda_r \circ H_B$$

for all $k \geq 0$. Note that the quasiconformal homeomorphism $\lambda_r \circ H_B : \Delta \to D_r$ can be homeomorphically extended to $\partial\Delta$ such that $(\lambda_r \circ H_B)(\partial\Delta) = \Gamma_r$. Since $\chi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a $K(d,\theta)$-quasiconformal homeomorphism, it follows that $\Gamma_r = \chi(\partial\Delta)$ is a $K(d,\theta)$-quasi-circle. This completes the proof of Lemma 3.6. □

To remove the condition that $P_f \cap D_r = \emptyset$ in Lemma 3.6, we need the following lemma.

Lemma 3.7 (Lemma 9.8 of [15]). *For any $C > 0$, there is a $1 < K(C) < \infty$ depending only on C such that for any simple closed curve $\gamma \subset \widehat{\mathbb{C}}$ if*

$$\left|\frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)}\right| > C \tag{12}$$

holds for any four points $\{w_1, w_2, w_3, w_4\}$ in γ which are listed according to anticlockwise order, then γ is a $K(C)$-quasi-circle. The converse is also true. That is, for any $1 < K < \infty$, there exists a $C(K) > 0$ depending only on K such that if $\gamma \subset \widehat{\mathbb{C}}$ is a K-quasi-circle, then for any four points $\{w_1, w_2, w_3, w_4\}$ in γ which are listed according to anticlockwise order, (12) holds with the constant in the right hand replaced by $C(K)$.

Let R_θ^d denote the set of all the degree d rational maps which have a fixed Siegel disk centered at the origin and with rotation number θ.

Lemma 3.8. *There is a $0 < C < \infty$ depending only on d and θ such that for any $f \in R_\theta^d$, any $0 < r < 1$, any four distinct integers k, l, m and n, and any $z \in \Gamma_r$, if $f^k(z), f^l(z), f^m(z)$ and $f^n(z)$ are ordered anticlockwise in Γ_r, then*

$$\left|\frac{(f^k(z) - f^m(z))(f^l(z) - f^n(z))}{(f^k(z) - f^n(z))(f^l(z) - f^m(z))}\right| > C.$$

Proof. Let $f \in R_\theta^d$ and D be the Siegel disk of f centered at the origin. Let $w \in \widehat{\mathbb{C}} \setminus D$. By considering the rational map $\frac{f(z)}{f(z)-w}$, we may assume that $\infty \notin D$. For $0 < r < 1$, let

$$V_f(r; k, l, m, n) = \inf_{z \in \Gamma_r} \left| \frac{(f^k(z) - f^m(z))(f^l(z) - f^n(z))}{(f^k(z) - f^n(z))(f^l(z) - f^m(z))} \right|$$

Note that as $z \to 0$, the function

$$C_{f;k,l,m,n}(z) = \frac{(f^k(z) - f^m(z))(f^l(z) - f^n(z))}{(f^k(z) - f^n(z))(f^l(z) - f^m(z))}$$

has a non-zero limit. We can thus regard $C_{f;k,l,m,n}$ as a holomorphic function in D which does not vanish. In particular, we have

$$V_f(r_1; k, l, m, n) \geq V_f(r_2; k, l, m, n)$$

for all $0 \leq r_1 < r_2 < 1$. It is important to note that $V_f(r; k, l, m, n)$ is preserved by an Möbius conjugation.

Let $\{f_i\}$ be a sequence in R_θ^d such that

$$\lim_{i \to \infty} V_{f_i}(r; k, l, m, n) = \inf_{f \in R_\theta^d} \{V_f(r; k, l, m, n)\}.$$

Let D_i denote the Siegel disk of f_i centered at the origin. Let Γ_r^i denote the Γ_r of D_i. For each i, take $a_i \in \Gamma_r^i$. By considering the sequence of rational maps $\frac{1}{a_i} f_i(a_i z)$ if necessary, we may assume that every Γ_r^i passes through the point 1.

For each i, let $\phi_i : \Delta \to D_i$ be the linearization map such that $\phi_i'(0) > 0$. Since every Γ_r^i passes through 1, it follows that $\phi_i'(0)$ is bounded away from 0 and the infinity. By Koebe's 1/4-Theorem, $D_i = \phi_i(\Delta)$ contains a Euclidean disk $B_0(\tau)$ for some $\tau > 0$. Since $f_i(0) = 0$ and $f_i'(0) = e^{2\pi i\theta}$, it follows that the sequence $\{f_i\}$ is normal in $B_0(\tau)$. By taking a convergent subsequence, we may assume that f_i converges to a univalent function g in $B_0(\tau)$. We claim that g is the restriction of some rational map to $B_0(\tau)$ whose degree is not more than d. Let us prove the claim. To this end, let us write

$$f_i(z) = c_i z \frac{\prod_k (z - p_k^i)}{\prod_j (z - q_j^i)}$$

where $c_i \neq 0$ and all the p_k^i and q_j^i do not belong to $B_0(\tau)$. By taking a subsequence we may assume that as $i \to \infty$, each of the p_k^i and q_j^i either converges to the infinity or converges to some complex number in the outside of $B_0(\tau)$. Let us denote this as $p_{k'}^i \to \infty$, $q_{j'}^i \to \infty$, $p_{k''}^i \to p_{k''}$, and $q_{j''}^i \to q_{j''}$ where the $p_{k''}$ and $q_{j''}$ are complex numbers in the outside of $B_0(\tau)$. Since when restricted to $B_0(\tau)$ f_i converges to g and $\frac{\prod_{k''}(z - p_{k''}^i)}{\prod_{j''}(z - q_{j''}^i)}$ converges to

$\frac{\prod_{k''}(z-p_{k''})}{\prod_{j''}(z-q_{j''})}$, it follows that as $i \to \infty$,

$$c_i \cdot \frac{\prod_{k'} p^i_{k'}}{\prod_{j'} q^i_{j'}} \to \alpha$$

where α is some nonzero complex number. This implies that in $B_0(\tau)$, the univalent function g is identified with the following rational function whose degree is clearly not more than d,

$$\alpha z \frac{\prod_{k''}(z-p_{k''})}{\prod_{j''}(z-q_{j''})}.$$

The claim has been proved. In the following let us still use g to denote this rational function.

By taking a convergent subsequence if necessary, we may assume that $\phi_i \to \phi$ uniformly in any compact subset of the unit disk where ϕ is some univalent function defined in the unit disk. In particular, in a small neighborhood of the origin, $g(z) = (\phi \circ R_\theta \circ \phi^{-1})(z)$ where R_θ is the rigid rotation given by θ. Since g is a rational map, it follows that

$$g(z) = (\phi \circ R_\theta \circ \phi^{-1})(z) \text{ for all } z \in \phi(\Delta). \tag{13}$$

Since $\phi_i \to \phi$ uniformly in any compact subset of the unit disk, f_i converges uniformly to g in any compact subset of $\phi(\Delta)$. There are three cases.

In the first case, g is a Möbius map. Since $g(0) = 0$ and $g'(0) = e^{2\pi i\theta}$, it follows that g has two distinct fixed points $\{0, p\}$, and moreover, $\widehat{\mathbb{C}} - \{0, p\}$ is foliated by g-invariant Euclidean circles. Since $\phi_i \to \phi$ uniformly in any compact subset of the unit disk, it follows that Γ^i_r converges to a Euclidean circle Γ which is preserved by g, and moreover, f_i uniformly converges to g in an open neighborhood of Γ. Since g is conjugate to the rigid rotation R_θ through a Möbius map, we thus have

$$\lim_{i\to\infty} V_{f_i}(r; k, l, m, n) = V_{R_\theta}(r; k, l, m, n).$$

The Lemma in this case then follows from Lemma 3.7 and the fact that the Euclidean circle is a quasi-circle.

In the second case, $g \in R^{d'}_\theta$ for some $2 \le d' < d$. Let D^g denote the Siegel disk of g centered at the origin. By (13), it follows that D^g always contains $\phi(\Delta)$ and may be strictly larger than $\phi(\Delta)$. Again since $\phi_i \to \phi$ uniformly in any compact subset of the unit disk, it follows that Γ^i_r converges to the $\Gamma_{r'}$ of D^g for some $0 < r' \le r$, and moreover, f_i uniformly converges to g in an open neighborhood of $\Gamma_{r'}$. This implies that

$$\lim_{i\to\infty} V_{f_i}(r; k, l, m, n) = V_g(r'; k, l, m, n) \ge V_g(r; k, l, m, n). \tag{14}$$

Since g is a rational map with degree less than d, by induction on the degree of the rational map we have a constant $0 < C < \infty$ depending only on d and

θ such that

$$V_g(r; k, l, m, n) > C.$$

Thus the Lemma also follows in this case.

In the third case, $g \in R_\theta^d$. Then we still have (14). Thus we get

$$V_g(r; k, l, m, n) = \inf_{f \in R_\theta^d} \{V_f(r; k, l, m, n)\}. \tag{15}$$

Recall that D^g denotes the Siegel disk of g centered at the origin. By a Möbius conjugation which preserves 0, we may assume that $\infty \notin D^g$ and $g(\infty) \neq \infty$. Let Γ_r^g and D_r^g denote the Γ_r and the D_r of D^g respectively. If $P_g \cap D_r^g = \emptyset$, then Γ_r^g is a $K(d, \theta)$-quasi-circle by Lemma 3.6. The Lemma in this case then follows from Lemma 3.7. Now suppose

$$P_g \cap D_r^g \neq \emptyset.$$

Let $V_1, \cdots, V_N$ denote all the components of $g^{-1}(D_r^g)$ in the outside of D^g such that

$$V_i \cap (\Omega_g \cup P_g) \neq \emptyset, \quad i = 1, \cdots, N.$$

For each $1 \le i \le N$, let

$$g(V_i \cap (\Omega_g \cup P_g)) = \{x_1, \cdots, x_{k_i}\}$$

where $k_i \geq 1$ is some integer. For each i, take k_i distinct points in Γ_r^g, say $z_1^i, \cdots, z_{k_i}^i$.

Now take an r' such that $r < r' < 1$. For each $1 \le i \le N$, take k_i disjoint Jordan domains with smooth boundaries, say $U_1^i, \cdots, U_{k_i}^i$ such that $\overline{U_j^i} \subset D_{r'}^g$ and $\{x_j^i, z_j^i\} \subset U_j^i$ for all $1 \le j \le k_i$, and most importantly,

$$d_{U_j^i}(x_j^i, z_j^i) \equiv C_0$$

holds for all $1 \le i \le N$ and $1 \le j \le k_i$, where $d_{U_j^i}(\cdot, \cdot)$ denotes the distance with respect to the hyperbolic metric in U_j^i. In fact, when the domain becomes thinner, the hyperbolic distance between the two points will become bigger. So it is easy to make all $d_{U_l^i}(x_j^i, z_j^i)$ taking the same large value by making all the domains U_j^i thin enough. It follows that there is a $t_0 \in \Delta$ such that for each U_j^i, there is a Riemann isomorphism

$$\psi_j^i : \Delta \to U_j^i$$

such that $\psi_j^i(0) = x_j^i$ and $\psi_j^i(t_0) = z_j^i$. Let ϕ_j^i denote the inverse of ψ_j^i. For each $1 \le i \le N$, define

$$\Phi_i(\cdot, \cdot) : D_{r'}^g \times \Delta \to D_{r'}^g$$

as follows

$$\Phi_i(z, t) = \begin{cases} z & \text{if } z \in D_{r'}^g \setminus \bigcup_{1 \le j \le k_i} U_j^i, \\ \psi_j^i\big(\phi_j^i(z) + (1 - |\phi_j^i(z)|)t\big) & \text{if } z \in U_j^i \text{ for some } 1 \le j \le k_i. \end{cases}$$

By a direct calculation, we have

$$\text{(16)} \quad \frac{(\Phi_i)_{\bar z}}{(\Phi_i)_z}(z,t) = \begin{cases} 0 & \text{if } z \in D^g_{r'} \setminus \bigcup_{1\le j\le k_i} U^i_j, \\ \frac{\overline{(\phi^i_j)'(z)}}{(\phi^i_j)'(z)} \frac{t\phi^i_j(z)}{\overline{t\phi^i_j(z)}-2|\phi^i_j(z)|} & \text{if } z \in U^i_j \text{ for some } 1 \le j \le k_i. \end{cases}$$

This implies that for almost every z in $D^g_{r'}$, the complex dilatation of Φ_i depends analytically on t when t varies in Δ. For each $1 \le i \le N$, let V_i' be the component of $g^{-1}(D^g_{r'})$ which contains V_i. Since V_i is in the outside of D^g, we have $V_i' \cap D^g = \emptyset$ for all $1 \le i \le N$. For each $t \in \Delta$, define

$$\text{(17)} \qquad h_t(z) = \begin{cases} g(z) & \text{if } z \in \widehat{\mathbb{C}} \setminus \bigcup_{1\le i\le N} V_i', \\ \Phi_i(g(z),t) & \text{if } z \in V_i' \text{ for some } 1 \le i \le N. \end{cases}$$

It follows that $h_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a branched covering map of degree d. Let

$$\Omega = \bigcup_{1\le i\le N} V_i' \cap \Big(\bigcup_{1\le j\le k_i} g^{-1}(U^i_j)\Big)$$

Since all the ∂U^i_j are smooth Jordan curves, $\partial\Omega$ is the union of finitely many quasi-circles. For each $t \in \Delta$, from (16) and (17) we can easily get

$$\frac{(h_t)_{\bar z}}{(h_t)_z}(z) = \begin{cases} 0 & \text{if } z \in \widehat{\mathbb{C}} \setminus \Omega, \\ \frac{\overline{(\phi^i_j)'(g(z))}}{(\phi^i_j)'(g(z))} \frac{t\phi^i_j(g(z))}{\overline{t\phi^i_j(g(z))}-2|\phi^i_j(g(z))|} \frac{\overline{g'(z)}}{g'(z)} & \text{if } z \in \Omega. \end{cases}$$

This implies that for every $t \in \Delta$, the map $h_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a quasi-regular branched covering map of degree d, and moreover, for almost every z, the complex dilatation of h_t at z depends analytically on t when t varies in Δ.

By the construction of h_t, it follows that for each $t \in \Delta$, $h_t|D^g = g|D^g$ is conformal in D^g, and moreover, for almost every $z \in \widehat{\mathbb{C}}$, if h_t is quasiconformal at some point z, then $h_t(z) \in D^g$. So for each $t \in \Delta$, by pulling back the standard complex structure μ_0 in the Siegel disk D^g through the iteration of h_t, we can get a h_t-invariant complex structure μ_t in the whole sphere. Again by a direct calculation we get

$$\mu_t(z) = \begin{cases} \frac{\overline{(g^n)'(z)}}{(g^n)'(z)} \frac{(h_t)_{\bar z}}{(h_t)_z}(g^n(z)) & \text{if } g^n(z) \in \Omega \text{ for some integer } n \ge 0, \\ 0 & \text{if otherwise.} \end{cases}$$

From the above formula, it follows that for almost every $z \in \widehat{\mathbb{C}}$, $\mu_t(z)$ depends analytically on t. Let ϕ_t be the quasiconformal homeomorphism of the sphere which fixes 0, 1 and the infinity and which solves the Beltrami equation given by μ_t. Then ϕ_t depends analytically on t. Let

$$g_t(z) = \phi_t \circ h_t \circ \phi_t^{-1}(z).$$

We claim

1. $g_0 = g$,

2. $g_t \in R_\theta^d$ for each $t \in \Delta$,
3. g_t depends analytically on t when t varies in Δ,
4. The post-critical set of g_{t_0} does not intersect the D_r of g_{t_0}.

Let us prove the claim. The first two assertions follow directly from the construction. Let us prove the third assertion (We would like to remark here that ϕ_t depends analytically on t does not imply that ϕ_t^{-1} depends analytically on t also). Note that $\infty \notin D^g$ and $g(\infty) \neq \infty$ by the assumption right behind (15). Take $p \in \mathbb{C}$ such that $p \notin D^g$ and $p \neq g(\infty)$. Let $a_1, \cdots, a_d$, counted by multiplicities, be all the p-value points of g, that is, $g(a_i) = p$ for $1 \leq i \leq d$. Let $b_i, 1 \leq i \leq d$, be all the poles of g, again counted by multiplicities. Since $\infty \notin D^g$ and $g(\infty) \neq \infty$, all the a_i and b_i are complex numbers. Since both p and ∞ do not belong to D^g, by the definition of h_t, it follows that $h_t(a_i) = g(a_i) = p$ and $h_t(b_i) = g(b_i) = \infty$ for all $t \in \Delta$ and $1 \leq i \leq d$. Then $\phi_t(a_i), 1 \leq i \leq d$, are all the $\phi_t(p)$-value points of g_t, and $\phi_t(b_i), 1 \leq i \leq d$, are all the poles of g_t. Since all the a_i and b_i do not belong to D^g, it follows that all the $\phi_t(a_i)$ and $\phi_t(b_i)$ do not belong to the Siegel disk of g_t centered at the origin, and thus are all non-zero complex numbers. Let

$$c(t) = \prod_{i=1}^{d} \frac{\phi_t(a_i)}{\phi_t(b_i)}.$$

Since $g_t(0) = 0$, it follows that

$$g_t(z) = \phi_t(p) - \frac{\phi_t(p)}{c(t)} \cdot \prod_{i=1}^{d} \frac{(z - \phi_t(a_i))}{(z - \phi_t(b_i))}.$$

This implies that g_t depends analytically on t. The third assertion follows. Now let us prove the last assertion. First note that $h_{t_0}|D^g = g|D^g$, and

$$(\Omega_{h_{t_0}} \cup P_{h_{t_0}}) - D^g = (\Omega_g \cup P_g) - D^g.$$

Suppose $z \in (\Omega_{h_{t_0}} \cup P_{h_{t_0}}) - D^g$ is a point such that $g(z) \in D^g$. By the previous construction it follows that $z \in V_i$ for some $1 \leq i \leq N$ and $g(z) = x_j^i$ for some $1 \leq j \leq k_i$. So $h_{t_0}(z) = \Phi_i(g(z), t_0) = z_j^i$ belongs to the Γ_r of D^g. Note that $\phi_{t_0} : D^g \to D^{g_{t_0}}$ is a holomorphic isomorphism and is the conjugation map between $g|D^g = h_{t_0}|D^g : D^g \to D^g$ and $g_{t_0}|D^{g_{t_0}} : D^{g_{t_0}} \to D^{g_{t_0}}$. So ϕ_{t_0} maps the Γ_r of D^g to the Γ_r of $D^{g_{t_0}}$. In particular, $g_{t_0}(\phi_{t_0}(z)) = \phi_{t_0}(z_j^i)$ belongs to the Γ_r of $D^{g_{t_0}}$. The last assertion of the claim has been proved. The proof of the claim is completed.

Now take z_0 in the Γ_r of D_g such that

$$|C_{g;k,l,m,n}(z_0)| = V_g(r; k, l, m, n).$$

Since for any given z, $\phi_t(z)$ is holomorphic in t for $t \in \Delta$, it follows that for every integer $i \geq 0$, the map $g_t^i(\phi_t(z_0)) = \phi_t(g^i(z_0))$ is holomorphic in t for

$t \in \Delta$. Thus the map

$$C_{g_t;k,l,m,n}(\phi_t(z_0)) = \frac{(g_t^k(\phi_t(z_0)) - g_t^m(\phi_t(z_0)))(g_t^l(\phi_t(z_0)) - g_t^n(\phi_t(z_0)))}{(g_t^k(\phi_t(z_0)) - g_t^n(\phi_t(z_0)))(g_t^l(\phi_t(z_0)) - g_t^m(\phi_t(z_0)))}$$

is a holomorphic function in t which does not vanish for $t \in \Delta$. Since ϕ_t maps the Γ_r of D^g to the Γ_r of D^{g_t}, $\phi_t(z_0)$ belong to the Γ_r of D^{g_t}. We thus have

$$|C_{g_t;k,l,m,n}(\phi_t(z_0))| \ge V_{g_t}(r;k,l,m,n) \text{ for all } t \in \Delta.$$

This, together with (15) and the choice of z_0, implies that the modulus of the holomorphic function $C_{g_t;k,l,m,n}(\phi_t(z_0))$ obtains the minimum at $t = 0$. Since $C_{g_t;k,l,m,n}(\phi_t(z_0))$ does not vanish for $t \in \Delta$, it follows that $C_{g_t;k,l,m,n}(\phi_t(z_0))$ is a constant function. In particular, we have

$$|C_{g;k,l,m,n}(z_0)| = |C_{g_{t_0};k,l,m,n}(\phi_{t_0}(z_0))| \ge V_{g_{t_0}}(r;k,l,m,n).$$

But by the last assertion of the claim we just proved, the postcritical set of g_{t_0} does not intersect the D_r of $D^{g_{t_0}}$. By Lemma 3.6 there is a $1 < C < \infty$ depending only on d and θ such that

$$V_{g_{t_0}}(r;k,l,m,n) > C.$$

This proves the lemma in the third case. The proof of Lemma 3.8 is completed. □

Now let us prove the Main Theorem. Since the forward orbit of any z in Γ_r is dense in Γ_r, it follows from Lemma 3.8 and Lemma 3.7 that there is a $1 < K(d,\theta) < \infty$ depending only on d and θ such that every Γ_r is a $K(d,\theta)$-quasi-circle. The Main Theorem then follow from Lemma 3.1.

4. Proof of Theorem B

4.1. **From Cross Ratios to Simple closed Geodesics.** For two distinct points $a, b \in \mathbb{T}$, let $[a,b]$ denote the arc segment which connects a and b in anti-clockwise direction. For an arc segment $I \subset \mathbb{T}$, let $|I|$ denote the length of I with respect to the Euclidean metric. We say an arc segment J is properly contained in I if $J \subset I$ and $I \setminus J$ consists of two non-trivial arc segments. In this case, we denote it by $J \Subset I$.

Now for any two arc segments $J \Subset I \subset \mathbb{T}$, we define

$$C(I,J) = \frac{|I||J|}{|R||L|}$$

where R and L denote the two arc components of $I - J$, respectively. From the definition, we have

Lemma 4.1. *Let $0 < K < \infty$. Then for any arc segments $J \Subset I \subset \mathbb{T}$, if $C(I,J) < K$, we have*

$$\min\{|R|,|L|\} > |J|/K.$$

By the above lemma, it follows that the value $C(I, J)$ measures the space around J in I.

Let $B \in \mathbf{B}_\theta^m$. Let $k \geq 1$ be an integer and $S, T \subset \mathbb{T}$ be two arc segments. We say S is the pull back of T by B^k if $B^k : S \to T$ is a homeomorphism. Suppose $J \Subset I \subset \mathbb{T}$ are two arc segments. Let us denote them by I_B^0 and J_B^0 respectively. For $k \geq 1$, let I_B^k and J_B^k denote the arc segments in $\mathbb{T}$ which are the pull backs of I_B^0 and J_B^0 respectively by B^k. The next lemma is the key in the proof of Theorem B.

Lemma 4.2. *Let $m = 2d - 1 \geq 3$ be an odd integer and $0 < \theta < 1$ be a bounded type irrational number. Then, there exist constants $\alpha \in (0, \infty)$ and $\beta \in (0, \infty)$ depending only on m and θ, such that for any centered Blaschke product $B \in \mathbf{B}_\theta^m$ and any disjoint family of arc segments $\{I_B^k \,|\, 0 \leq k \leq N\}$ and any family of arc segments $\{J_B^k \,|\, 0 \leq k \leq N\}$ with $J_B^k \Subset I_B^k$ for all $0 \leq k \leq N$, we have*

$$C(I_B^N, J_B^N) \leq \beta \cdot (1 + C(I_B^0, J_B^0))^\alpha.$$

The main task in the proof of Theorem A in [9] is to prove that the Światek distortion has a uniform upper bound for all the Blaschke products in $\mathbf{H}_\theta^m$. The difference between the two situations is that Herman's proof uses real techniques and relies essentially on the compact property of $\mathbf{H}_\theta^m$, which does not hold for $\mathbf{B}_\theta^m$ (see Lemma 2.1 and Remark 1). To solve this problem, we make use of the complex analytic property of the maps in $\mathbf{B}_\theta^m$. Instead of considering cross ratios, we consider the length of certain simple closed geodesics. As a result, we reduce Lemma 4.2 to showing that the length of certain simple closed geodesics, after disjoint pull backs, can be increased by at most some factor which is bounded above by a constant depending only on m (Lemma 4.3). Let us introduce some notations before we expose this idea further.

Let $\widehat{\mathbb{C}}$ denote the Riemann sphere. Let $B \in \mathbf{B}_\theta^m$ be a centered Blaschke product. For $0 \leq k \leq N$, let I_B^k and J_B^k be the arc segments given in Lemma 4.2. Let

$$X_B^k = (\widehat{\mathbb{C}} - \mathbb{T}) \cup (I_B^k - J_B^k)$$

Then there exists a unique simple closed geodesic in X_B^k which separates J_B^k and $\mathbb{T} - I_B^k$. Let us denote it by γ_B^k. Let $l_{X_B^k}(\gamma_B^k)$ denote the length of γ_B^k with respect to the hyperbolic metric in X_B^k. The goal of this section is to reduce Lemma 4.2 to the following lemma.

Lemma 4.3. *Let $m = 2d - 1 \geq 3$ be an odd integer and $0 < \theta < 1$ be a bounded type irrational number. Then there exists a $1 < C(m) < \infty$ which depends only on m such that for any Blaschke product $B \in \mathbf{B}_\theta^m$, and any disjoint family of arc segments $\{I_B^k \,\big|\, 0 \leq k \leq N\}$ and any family of arc* *segments* $\{J_B^k \,|\, 0 \le k \le N\}$ *with* $J_B^k \Subset I_B^k$ *for all* $0 \le k \le N$*, we have*

$$\frac{l_{X_B^N}(\gamma_B^N)}{l_{X_B^0}(\gamma_B^0)} \le C(m).$$

Proposition 4.1. Lemma 4.3 implies Lemma 4.2.

We need to prove Lemmas 4.4-4.7 before we prove Proposition 4.1. For $T \in (0, \infty)$, let $\Lambda(T)$ be the modulus of the annulus $\mathbb{C} \setminus ([-1, 0] \cup [T, \infty))$.

Lemma 4.4. *For all* $T \in (0, \infty)$*, we have*

$$\Lambda(T) \cdot \Lambda(1/T) = 1/4 \ \text{ and } \ T < e^{2\pi\Lambda(T)} \le 16(T + 1).$$

Proof. See Chapter III of [1]. □

Lemma 4.5. *Let* $A \subset \widehat{\mathbb{C}}$ *be an annulus and* $\gamma \subset A$ *be its core geodesic. Then*

$$l_A(\gamma) = \frac{\pi}{\operatorname{mod}(A)}$$

where $l_A(\gamma)$ *is the length of* γ *with respect to the hyperbolic metric in* A*.*

Proof. We may assume that A is a Euclidean annulus $\{z \,|\, e^{-\alpha} < |z| < e^{\alpha}\}$ for some $\alpha > 0$. It follows that

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log \frac{e^{\alpha}}{e^{-\alpha}} = \frac{\alpha}{\pi}.$$

To compute the length of the core geodesic γ of A, consider the vertical strip

$$S = \{z = x + iy \,|\, -\alpha < x < \alpha, -\infty < y < +\infty\}.$$

The map $\Phi : z \mapsto e^z$ is a holomorphic covering map from S to A. Let $\Gamma = [-\pi i, \pi i]$ be the vertical straight segment. It is clear that $l_S(\Gamma) = l_A(\gamma)$. To compute $l_S(\Gamma)$, let us consider the map

$$\Psi : w \mapsto e^{i\frac{\pi}{2\alpha}w}.$$

The map Ψ maps S isomorphically to the right half plane H. Under this map, the vertical straight segment Γ is mapped to the horizontal straight segment $\Gamma' = [e^{-\frac{\pi^2}{2\alpha}}, e^{\frac{\pi^2}{2\alpha}}]$. We thus have

$$l_A(\gamma) = l_S(\Gamma) = l_H(\Gamma') = \int_{e^{-\frac{\pi^2}{2\alpha}}}^{e^{\frac{\pi^2}{2\alpha}}} \frac{1}{x} dx = \frac{\pi^2}{\alpha} = \frac{\pi}{\operatorname{mod}(A)}.$$

This completes the proof of Lemma 4.5. □

Lemma 4.6. *For any arc segments* $J \Subset I \subset \mathbb{T}$*, we have*

$$\frac{(2\pi - |I|)^2}{4\pi^2} \cdot C(I, J) \le e^{l_X(\gamma)/2} \le 4\pi^2 \cdot (1 + C(I, J)),$$

where $X = (\widehat{\mathbb{C}} - \mathbb{T}) \cup (I - J)$ *and* $l_X(\cdot)$ *denotes the length with respect to the hyperbolic in* X*.*

Proof. Assume that $I = [e^{i\theta_1}, e^{i\theta_4}]$ and $J = [e^{i\theta_2}, e^{i\theta_3}]$ and assume that $0 \le \theta_1 < \theta_2 < \theta_3 < \theta_4 \le 2\pi$. Let M be the Möbius transformation sending $e^{i\theta_2}$ to 0, $e^{i\theta_3}$ to -1, and $e^{i\theta_4}$ to ∞. Then $M(e^{i\theta_1}) \in (0, +\infty)$. Let $T = 1/M(e^{i\theta_1})$. By Lemmas 4.4 and 4.5 it follows that

$$l_X(\gamma) = \frac{\pi}{\Lambda(1/T)} = 4\pi\Lambda(T).$$

This, together with the second inequality of Lemma 4.4, implies

$$T < e^{2\pi\Lambda(T)} = e^{l_X(\gamma)/2} \le 16(T+1).$$

Since the cross ratio is preserved by Möbius transformation, it follows that

$$T = \left|\frac{(e^{i\theta_3} - e^{i\theta_2})(e^{i\theta_4} - e^{i\theta_1})}{(e^{i\theta_4} - e^{i\theta_3})(e^{i\theta_1} - e^{i\theta_2})}\right|$$

Since $|I| = \theta_4 - \theta_1$, $|J| = \theta_3 - \theta_2$, $|R| = \theta_4 - \theta_3$ and $|L| = \theta_2 - \theta_1$, we have

$$T = \left|\frac{(e^{i|I|} - 1)(e^{i|J|} - 1)}{(e^{i|R|} - 1)(e^{i|L|} - 1)}\right| = \left|\frac{\sin(|I|/2)\sin|J|/2)}{\sin|R|/2)\sin(|L|/2)}\right|. \tag{18}$$

Note that for $x \in (0, 2\pi)$, we have $4\pi\sin(x/2) \ge x(2\pi - x)$ and $0 \le \sin(x/2) \le x/2$. Both the inequalities can be easily proved by calculus and we shall leave the proofs to the reader. From these two inequalities and (18) we get

$$T \ge \frac{1}{4\pi^2}\frac{|I|(2\pi - |I|)\cdot|J|(2\pi - |J|)}{|L|\cdot|R|} \ge \frac{1}{4\pi^2}\cdot C(I,J)\cdot(2\pi - |I|)^2.$$

Since $T < e^{l_X(\gamma)/2}$, it follows that

$$\frac{(2\pi - |I|)^2}{4\pi^2}\cdot C(I,J) \le e^{l_X(\gamma)/2}. \tag{19}$$

Note that for $x \in [0, \pi]$, we have $x/\pi \le \sin(x/2) \le x/2$. Again the inequality can be easily proved by calculus and we omit the proof here. Thus, if $|L| \le \pi$ and $|R| \le \pi$, from this inequality and (18) we get

$$T \le \frac{\pi^2}{4}C(I,J).$$

If $\pi \le |L| \le |I|$, then $|R| \le \pi$ and

$$T \le \frac{\sin(|J|/2)}{\sin(|R|/2)} \le \frac{\pi}{2}\frac{|J|}{|R|} \le \frac{\pi}{2}C(I,J).$$

If $\pi \le |R| \le |I|$, then $|L| \le \pi$ and

$$T \le \frac{\sin(|J|/2)}{\sin(|L|/2)} \le \frac{\pi}{2}\frac{|J|}{|L|} \le \frac{\pi}{2}C(I,J).$$

In all the cases we have

$$e^{l_X(\gamma)/2} \le 16(T+1) \le 4\pi^2 C(I,J) + 16 < 4\pi^2\cdot(1 + C(I,J)). \tag{20}$$

Lemma 4.6 then follows from (19) and (20). $\square$

For any $B \in \mathbf{B}_\theta^m$, recall that μ_B is the invariant probability measure of $B|\mathbb{T} : \mathbb{T} \to \mathbb{T}$.

Lemma 4.7. *Assume that* $B \in \mathbf{B}_\theta^m$ *is centered and* $I \subset \mathbb{T}$ *is an arc segment such that* $\mu_B(I) < \delta \le 1/2$. *Then*

$$|\mathbb{T} - I| \ge 2 \arccos \frac{\delta}{1-\delta}.$$

Proof. Set $\eta = \mu_B(I)$. Then $\eta \le \delta$ and $\mu_B(\mathbb{T} - I) = 1 - \eta$. Set $L = |\mathbb{T} - I|$ and without loss of generality, let us assume that $\mathbb{T} - I = [e^{-L/2}, e^{L/2}]$ is the arc segment in $\mathbb{T}$ which connects $e^{-L/2}$ and $e^{L/2}$ anticlockwise. Since $0 \le L/2 \le \pi$ and the function $x \mapsto \cos(x)$ is decreasing on $[0, \pi]$, it follows that for every $z \in \mathbb{T} - I$, one has

$$\Re(z) \ge \cos(L/2).$$

It is clear that $\Re(z) \ge -1$ for all $z \in I$. Since B is centered, by (3) we have $\int_{\mathbb{T}} z d\mu_B(z) = 0$. We thus get

$$\int_{\mathbb{T}} \Re(z) d\mu_B(z) = \Re\big(\int_{\mathbb{T}} z d\mu_B(z) \big) = 0.$$

Since

$$\int_{\mathbb{T}} \Re(z) d\mu_B(z) = \int_{\mathbb{T}-I} \Re(z) d\mu_B(z) + \int_I \Re(z) d\mu_B(z) \ge (1-\eta)\cos(L/2) - \eta,$$

we have

$$(1-\eta)\cos(L/2) - \eta \le 0.$$

This implies that

$$\cos(L/2) \le \frac{\eta}{1-\eta} \le \frac{\delta}{1-\delta}$$

and thus

$$L \ge 2 \arccos \frac{\delta}{1-\delta}.$$

Lemma 4.7 follows. $\square$

Now it is the time to prove Proposition 4.1.

Proof. If $N = 0$, the result is trivial. So let us assume that $N \ge 1$. Since I_B^N is disjoint from I_B^{N-1}, we have that

$$\mu_B(I_B^N) \le \delta = \min\{\theta, 1-\theta\} < 1/2.$$

According to Lemma 4.7, we have

$$2\pi - |I_B^N| = |\mathbb{T} - I_B^N| \ge \epsilon = 2 \arccos \frac{\delta}{1-\delta}.$$

According to Lemma 4.6, we have

$$C(I_B^N, J_B^N) \le \frac{4\pi^2}{\epsilon^2} e^{l_{X_B^N}(\gamma_B^N)/2} \le \frac{4\pi^2}{\epsilon^2} e^{\alpha \cdot l_{X_B^0}(\gamma_B^0)/2}$$

where $\alpha = C(m)$ is the constant provided by Lemma 4.3. The result then follows by taking $\beta = \epsilon^{-2} \cdot (4\pi^2)^{1+\alpha}$ since by Lemma 4.6, we have

$$e^{\alpha \cdot l_{X_B^0}(\gamma_B^0)/2} \le (4\pi^2)^\alpha (1 + C(I_B^0, J_B^0))^\alpha.$$

This completes the proof of Proposition 4.1. $\square$

4.2. **Proof of Lemma 4.3.** The proof of Lemma 4.3 is based on Lemmas 4.8-4.13. Before we state and prove these lemmas, let us introduce some common notations which will be used in all these lemmas. Let $N \ge 1$ be an arbitrary integer. Let $J^k \Subset I^k \subset \mathbb{T}$, $0 \le k \le N$, be arc segments such that all $I^k, 0 \le k \le N$, are disjoint with each other. Let $p \ge 1$ be an integer and $Z = \{z_1, \cdots, z_p\}$ be a finite subset of $\widehat{\mathbb{C}}$ containing p points. For $0 \le k \le N$, we set

$$U_k = (\widehat{\mathbb{C}} - \mathbb{T}) \cup (I^k - J^k) \quad \text{and} \quad V_k = U_k - Z.$$

We let l_k be the length of the core geodesic of the annulus U_k and l_k' be the length of a shortest simple closed geodesic in V_k separating J^k and $\mathbb{T} - I^k$ (there may be several geodesics with minimal length). Note that $l_k \le l_k'$.

Lemma 4.8. *Let A be an annulus and $Z = \{z_1, \cdots, z_p\} \subset A$. Then, there is an annulus $B \subset A \setminus Z$ homotopic to A with*

$$\mathrm{mod}(A) \le (p+1)\mathrm{mod}(B).$$

Proof. Without loss of generality, we may assume that A is a round annulus $\{z \,|\, r < |z| < R\}$ for some $0 \le r < R$. Cutting A along at most p round circles passing through the points in Z, we find at most $p+1$ round annuli contained in $A - Z$, whose moduli add up to that of A. Let B be one of those subannuli with maximal modulus. Then $\mathrm{mod}(A) \le (p+1)\mathrm{mod}(B)$. This completes the proof of Lemma 4.8. $\square$

Corollary 4.1. For all $0 \le k \le N$, we have $l_k' \le (p+1) \cdot l_k$.

Proof. Apply Lemma 4.8 with $A = U_k$ and obtain an annulus $B \subset U_k - Z = V_k$ homotopic to A such that $\mathrm{mod}(A) \le (p+1)\mathrm{mod}(B)$. This implies that

$$\mathrm{mod}(U_k) \le (p+1)\mathrm{mod}(B).$$

Let γ_k' be the core geodesic of B. Then by Lemma 4.5 we have

$$l_k' \le l_{V_k}(\gamma') \le l_B(\gamma_k') = \frac{\pi}{\mathrm{mod}(B)} \le (p+1)\frac{\pi}{\mathrm{mod}(U_k)} = (p+1) \cdot l_k.$$

$\square$

Definition 4.1. Let $I \subset \mathbb{T}$ be an arc segment. Let Γ be the unique Euclidean circle which passes through the end points of I and is orthogonal to the unit circle (In the case that $|I| = \pi$, Γ is a straight line). We use $D(I)$ to denote the component of $\widehat{\mathbb{C}} - \Gamma$ which contains the interior of I.

Remark 2. From the definition, it is clear that if $|I| < \pi$, $D(I)$ is a Euclidean disk; if $|I| = \pi$, $D(I)$ is a half plane; and if $|I| > \pi$, $D(I)$ is the outside of a Euclidean disk.

Lemma 4.9. *Suppose that $J \Subset I$ are two arc segments. Let γ be the core geodesic of $(\widehat{\mathbb{C}} - \mathbb{T}) \cup (I - J)$. Then γ is a Euclidean circle orthogonal to the unit circle. In particular, $\gamma \subset D(I)$.*

Proof. Let $I = [a, d]$ and $J = [b, c]$. Let ϕ be a Möbius map which maps a to ∞, b to -1 and c to 0. Then ϕ maps d to some point $T \in (0, +\infty)$ and maps the unit circle to the real line. Let Γ be the Euclidean circle with center -1 and radius $\sqrt{1+T}$. Note that $\mathbb{C} - ([-1, 0] \cup [T, \infty))$ is symmetric about Γ.

Let Ω be the disk $\{z \,|\, |z+1| < \sqrt{1+T}\}$. Let $H = \Omega \setminus [-1, 0]$. Let $0 < r < 1$ be the number such that

$$\mathrm{mod}(H) = \frac{1}{2\pi} \log \frac{1}{r}.$$

Let $\psi : H \to \{z \mid r < |z| < 1\}$ be the holomorphic isomorphism such that the outer boundary component of H is mapped to the unit circle. Then by Schwarz Reflection Lemma the map ψ can be extended to a holomorphic isomorphism between $\mathbb{C} - ([-1, 0] \cup [T, \infty))$ and the annulus $\{z | r < |z| < r^{-1}\}$. In particular, ψ maps Γ to the unit circle which is the core geodesic of the annulus $\{z \mid r < |z| < r^{-1}\}$. This implies that Γ is the core geodesic of $\mathbb{C} - ([-1, 0] \cup [T, \infty))$. This implies that $\phi^{-1}(\Gamma)$, which must be a Euclidean circle orthogonal to the unit circle, is the core geodesic of $(\widehat{\mathbb{C}} - \mathbb{T}) \cup (I - J)$. The proves the first assertion. The second assertion follows directly from the first assertion and the definition of $D(I)$. This completes the proof of Lemma 4.9. □

For $z \in \widehat{\mathbb{C}} \setminus \mathbb{T}$, let ϕ_z be a Möbius map sending z to 0 and preserving $\mathbb{T}$. It is clear that ϕ_z is unique up to a post-composition with a rotation. For an arc segment $I \subset \mathbb{T}$, set

$$\mu_z(I) = |\phi_z(I)|.$$

Definition 4.2. Let $z \in \widehat{\mathbb{C}}$ and $I \subset \mathbb{T}$ be an arc segment. We say that z is in the shadow of I or shadowed by I if either $z \in I$ or if $z \in \widehat{\mathbb{C}} \setminus \mathbb{T}$ with $\mu_z(I) \geq 2\pi/3$.

The following lemma can be directly derived from the definitions and the reader shall easily provide a proof.

Lemma 4.10. *Let $z \in \widehat{\mathbb{C}}$ and $I \subset \mathbb{T}$ be an arc segment. Then the following three properties hold,*

1. *$z \in D(I)$ if and only if $z \in I$ or $\mu_z(I) > \pi$,*
2. *if $z \in D(I)$, then z is in the shadow of I,*
3. *z can be shadowed by at most three disjoint arc segments.*

For a hyperbolic Riemann surface X, we use ρ_X to denote the hyperbolic metric in X and $d_x(\cdot, \cdot)$ denote the distance with respect to the hyperbolic metric ρ_U.

Lemma 4.11. *For any $d_0 > 0$, there exists a $0 < C_0 < \infty$ depending only on d_0 such that for any two distinct points $x, y \in \Delta$, the inequality*

$$\frac{\rho_{\Delta-\{y\}}(x)}{\rho_\Delta(x)} \le 1 + C_0 e^{-2d_\Delta(x,y)}$$

holds provided that $d_\Delta(x, y) > d_0$.

Proof. We need only to show that C_0 can be taken to be a fixed constant when $d_\Delta(x, y)$ is large enough. To show this, it is sufficient to consider the case that $y = 0$ and $x = 1 - \delta$ with $0 < \delta < 1$ small. By direct calculations, we have

$$\rho_{\Delta-\{y\}}(x) = \frac{1}{(1-\delta)|\ln(1-\delta)|} \quad \text{and} \quad \rho_\Delta(x) = \frac{1}{\delta(1-\delta/2)}.$$

Note that for all $0 < \delta < 1$, we have

$$(1-\delta)|\ln(1-\delta)| > (1-\delta)(\delta + \delta^2/2 + \delta^3/3) > \delta(1 - \delta/2 - \delta^2)$$

and for all $0 < \delta < 1/2$, we have

$$\delta/2 + \delta^2 < 1/2.$$

Thus for all $0 < \delta < 1/2$, we have

$$\frac{\rho_{\Delta-\{y\}}(x)}{\rho_\Delta(x)} < 1 + \frac{\delta^2}{1 - \delta/2 - \delta^2} < 1 + 2\delta^2.$$

By a direct calculation, we get

$$d_\Delta(x, y) = \ln \frac{2-\delta}{\delta}.$$

The lemma then follows since

$$e^{-2d_\Delta(x,y)} = \frac{\delta^2}{(2-\delta)^2} > \delta^2/4.$$

□

Lemma 4.12. *There is a universal constant $0 < C < \infty$ such that for any arc segment $I \subset \mathbb{T}$ with $|I| < 2\pi/3$, we have*

$$\frac{\rho_{W-\{0\}}}{\rho_W} \le e^{C|I|} \text{ on } D(I)$$

where $W = \widehat{\mathbb{C}} - (\mathbb{T} - I)$.

Proof. For $0 < \alpha < \pi$, let

$$D_\alpha(I) = \{z \in W \mid d_W(z, I) < \ln \cot \frac{\alpha}{4}\}. \tag{21}$$

By transforming the unit circle to the real line through a Möbius map, it follows that D_α is the hyperbolic neighborhood of I with the exterior angle being α. More precisely, D_α is a simply connected domain containing I whose boundary is the union of two arc segments of Euclidean circles which are symmetric about the unit circle such that the exterior angle between ∂D_α and the unit circle is α. To learn more details about the hyperbolic neighborhood in a slit plane, we refer the reader to [13] (§5 of Chapter VI). By the definition of $D(I)$, we get

$$D(I) = D_{\pi/2}(I) = \{z \in W \mid d_W(z, I) < \ln \cot \frac{\pi}{8}\}.$$

It is not difficult to see that $0 \in \partial D_{|I|/2}(I)$. So we have

$$d_W(0, D(I)) = \ln \cot \frac{|I|}{8} - \ln \cot \frac{\pi}{8}.$$

Since $|I| \le 2\pi/3$, we have $0 < \sin \frac{|I|}{8} < |I|/8$. We thus get

$$\ln \cot \frac{|I|}{8} > \ln \frac{\cos \frac{\pi}{12}}{\frac{|I|}{8}} = \ln \frac{8 \cos \frac{\pi}{12}}{|I|}.$$

So for any $z \in D(I)$, we have

$$d_W(0, z) > d_W(0, D(I)) \ge \ln \frac{8 \cos \frac{\pi}{12}}{|I|} - \ln \cot \frac{\pi}{8}. \tag{22}$$

Since $|I| \le 2\pi/3$, we have $\cot \frac{|I|}{8} > \cot \frac{\pi}{12}$ and thus

$$d_W(0, z) > d_W(0, D(I)) \ge d_0 = \ln \cot \frac{\pi}{12} - \ln \cot \frac{\pi}{8} > 0. \tag{23}$$

For such d_0, let C_0 be the constant provided by Lemma 4.11. Then for any $z \in D(I)$, by Lemma 4.11 and (22), we have

$$\frac{\rho_{W-\{0\}}(z)}{\rho_W(z)} \le 1 + C_0 e^{-2d_W(0,z)} < 1 + \frac{C_0 \cdot \cot^2 \frac{\pi}{8}}{64 \cos^2 \frac{\pi}{12}} |I|^2.$$

Since $|I| < 2\pi/3$, we have $|I|^2 < \frac{2\pi}{3}|I|$. Take

$$C = \frac{\pi \cdot C_0 \cdot \cot^2 \frac{\pi}{8}}{96 \cos^2 \frac{\pi}{12}}.$$

We then have for any $z \in D(I)$,

$$\frac{\rho_{W-\{0\}}(z)}{\rho_W(z)} \le 1 + C|I| < e^{C|I|}.$$

The proof of Lemma 4.12 is completed. □

Relative Schwarz Lemma ([2]). *Let R and S be two hyperbolic Riemann surfaces and $f : R \to S$ be a holomorphic map. Then*

$$\frac{f^*\rho_{_S}}{\rho_{_R}} \le \frac{f^*\rho_{_{S'}}}{\rho_{_{R'}}} \le 1.$$

For a detailed proof of the Relative Schwarz Lemma, we refer the reader to [2].

Lemma 4.13. *Let C be the universal constant provided by Lemma 4.12. Let $J \Subset I \subset \mathbb{T}$ be two arc segments and $Z \subset \widehat{\mathbb{C}}$ be a finite set such that no point in Z is shadowed by I. Let γ be the core geodesic of the annulus $U = (\widehat{\mathbb{C}} - \mathbb{T}) \cup (I - J)$. Then*

$$\frac{l_{U-Z}(\gamma)}{l_U(\gamma)} \le \prod_{z \in Z} e^{C\mu_z(I)}.$$

Proof. Let $V = \widehat{\mathbb{C}} - (\mathbb{T} - I)$. Let us label the points in Z by $z_1, \cdots, z_p$. Let $Z_0 = \emptyset$ and for $1 \le k \le p$, let $Z_k = \{z_1, \cdots, z_k\}$. Note that

$$\frac{\rho_{U-Z}}{\rho_U} = \prod_{k=0}^{p-1} \frac{\rho_{U-Z_{k+1}}}{\rho_{U-Z_k}}.$$

It follows from the Relative Schwarz Lemma that

$$\frac{\rho_{U-Z_{k+1}}}{\rho_{U-Z_k}} \le \frac{\rho_{U-\{z_{k+1}\}}}{\rho_U} \le \frac{\rho_{V-\{z_{k+1}\}}}{\rho_V}.$$

So we finally have

$$\frac{\rho_{U-Z}}{\rho_U} \le \prod_{z \in Z} \frac{\rho_{V-\{z\}}}{\rho_V}. \tag{24}$$

Let ϕ_z be a Möbius map which preserves the unit circle and maps z to 0. Then $\phi_z(D(I)) = D(\phi_z(I))$. Since z is not shadowed by I, we have $|\phi_z(I)| < 2\pi/3$. Note that $\phi_z(V) = \widehat{\mathbb{C}} - (\mathbb{T} - \phi_z(I))$. By Lemma 4.12, we have

$$\frac{\rho_{\phi_z(V)-\{0\}}}{\rho_{\phi_z(V)}} \le e^{C|\phi_z(I)|} = e^{C\mu_z(I)} \text{ on } D(\phi_z(I)) = \phi_z(D(I)).$$

Since the maps $\phi_z : V \to \phi_z(V)$ and $\phi_z : V - \{z\} \to \phi_z(V) - \{0\}$ are holomorphic isomorphisms, it follows that

$$\frac{\rho_{V-\{z\}}(w)}{\rho_V(w)} = \frac{\rho_{\phi_z(V)-\{0\}}(\phi_z(w))}{\rho_{\phi_z(V)}(\phi_z(w))} \le e^{C\mu_z(I)} \text{ for all } w \in D(I).$$

This implies that

$$\frac{\rho_{V-\{z\}}}{\rho_V} \le e^{C\mu_z(I)} \text{ on } D(I) \tag{25}$$

From (24) and (25) we have

$$\frac{\rho_{U-Z}}{\rho_U} \le \prod_{z\in Z} e^{C\mu_z(I)} \text{ on } D(I).$$

Note that $\gamma \subset D(I)$ by Lemma 4.9. We thus have

$$\frac{\rho_{U-Z}}{\rho_U} \le \prod_{z\in Z} e^{C\mu_z(I)} \text{ on } \gamma.$$

Lemma 4.13 then follows. $\square$

Now let us prove Lemma 4.3.

Proof. Let $B \in \mathbf{B}_\theta^m$. In the beginning of §4.2, let $I^k = I_B^k$ and $J^k = J_B^k$ where $J_B^k \Subset I_B^k \subset \mathbb{T}$, $0 \le k \le N$, are the arc segments given in Lemma 4.3. for $0 \le k \le N$. Let Z be the set of all the critical values of B and $p = \#Z$. Then

$$U_k = X_B^k, V_k = X_B^k - Z \text{ and } l_k = l_{X_B^k}(\gamma_B^k) \text{ for } 0 \le k \le N.$$

Since the number of critical values of B is not more than the number of distinct critical points of B which is not more than $2m-2$, it follows that $p \le 2m-2$.

Let

$$\Lambda_1 = \{0 \le k \le N-1 \mid I_k \text{ shadows at least one point of } Z\}.$$

By the third assertion of Lemma 4.10, each point in Z is shadowed by at most three intervals I^k. This implies that

$$|\Lambda_1| \le 3p \le 6(m-1).$$

Let

$$\Lambda_2 = \{0 \le k \le N-1 \mid k \notin \Lambda_1\}.$$

Then

$$\frac{l_{X_B^N}(\gamma_B^N)}{l_{X_B^0}(\gamma_B^0)} = \frac{l_N}{l_0} = \prod_{k=0}^{N-1} \frac{l_{k+1}}{l_k} = \left(\prod_{k\in\Lambda_1} \frac{l_{k+1}}{l_k}\right) \cdot \left(\prod_{k\in\Lambda_2} \frac{l_{k+1}}{l_k}\right).$$

Claim 1.

$$\frac{l_{k+1}}{l_k} \le m(2m-1) \text{ for every } k \in \Lambda_1. \tag{26}$$

Let us prove the Claim 1. Let $k \in \Lambda_1$. Let ξ_B^k be one of the shortest simple closed geodesics in V_k separating J^k and $\mathbb{T} - I^k$. By the minimal property of ξ_B^k, it follows that ξ_B^k is symmetric about the unit circle. In particular, the unit circle and ξ_B^k have two intersection points where they cross perpendicularly. Let a_k and b_k be the two intersection points. Let a_k' and b_k' be the two points in the unit circle such that $B(a_k') = a_k$ and $B(b_k') = b_k$. Let W_{k+1} be the component of $B^{-1}(V_k)$ which contains a_k'. It is clear that $W_{k+1} \subset U_{k+1}$ and the map $B : W_{k+1} \to V_k$ is a holomorphic covering map. Let η_B^{k+1} be the simple closed geodesic in W_{k+1} such that $a_k' \in \eta_B^{k+1}$ and $B(\eta_B^{k+1}) = \xi_B^k$. Then

η_B^{k+1} crosses the unit circle at a'_k perpendicularly. It follows that η_B^{k+1} and the unit circle must have at least two intersection points. Since ξ_B^k intersects the unit circle at exactly two points a_k and b_k and the map $B|\mathbb{T}:\mathbb{T}\to\mathbb{T}$ is a homeomorphism, η_B^{k+1} and the unit circle have exactly two intersection points, a'_k and b'_k. Since ξ_B^k crosses the unit circle perpendicularly, η_B^{k+1} crosses the unit circle perpendicularly also. In particular, η_B^{k+1} separates $\mathbb{T}-I^{k+1}$ and J^{k+1}. Thus we have

$$l_{X_B^{k+1}}(\gamma_B^{k+1}) \le l_{X_B^{k+1}}(\eta_B^{k+1}).$$

Since $W_{k+1}\subset U_{k+1}$ we have $\rho_{W_{k+1}} \ge \rho_{U_{k+1}}$. So we have

$$l_{X_B^{k+1}}(\eta_B^{k+1}) \le l_{W_{k+1}}(\eta_B^{k+1}).$$

Since $B: W_{k+1}\to V_k$ is a holomorphic covering map and the degree of B is m, it follows that

$$l_{W_{k+1}}(\eta_B^{k+1}) \le m\cdot l_{V_k}(\xi_B^k).$$

By the choice of ξ_B^k and Corollary 4.1, we have

$$l_{V_k}(\xi_B^k) = l'_k \le (p+1)\cdot l_{U_k}(\gamma_B^k) = (p+1)\cdot l_k \le (2m-1)\cdot l_k.$$

This, together with the above three inequalities, implies that

$$l_{k+1} = l_{X_B^{k+1}}(\gamma_B^{k+1}) \le m(2m-1)\cdot l_k.$$

This proves (26) and the Claim 1 has been proved.

Let $0<C<\infty$ be the universal constant in Lemma 4.13.

Claim 2.

$$\frac{l_{k+1}}{l_k} \le \prod_{z\in Z} e^{C\mu_z(I^k)} \text{ for every k} \in \Lambda_2. \tag{27}$$

Let us prove the Claim 2. Let $k\in\Lambda_2$. By Lemma 4.9, we have $\gamma_B^k\subset D(I^k)$. Since I^k does not shadow any point in Z, it follows that $D(I^k)$ does not intersect Z. This implies that γ_B^k does not contain any point in Z. We thus have $\gamma_B^k\subset V_k$. Let ξ_B^k be the unique simple closed geodesic in V_k which is homotopic to γ_B^k in V_k. Then ξ_B^k separates $\mathbb{T}-I^k$ and J^k, and moreover,

$$l_{V_k}(\xi_B^k) \le l_{V_k}(\gamma_B^k). \tag{28}$$

Since γ_B^k and V_k are symmetric about the unit circle, ξ_B^k is symmetric about the unit circle also. In particular, the unit circle and ξ_B^k have two intersection points where they cross perpendicularly. Now let W_{k+1} and η_B^{k+1} be as in the proof of the Claim 1. By the same argument as before, it follows that η_B^{k+1} separates $\mathbb{T}-I^{k+1}$ and J^{k+1}, and the map $B: W_{k+1}\to V_k$ is a holomorphic covering map. Let Ω be the component of $\widehat{\mathbb{C}}-\gamma_B^k$ which contains J^k. Since

$D(I^k)$ does not intersect the set Z and since $\gamma_B^k \subset D(I^k)$ by Lemma 4.9, it follows that

$$\Omega \cap Z = \emptyset. \tag{29}$$

Let $\tilde{\Omega}$ be the component of $\widehat{\mathbb{C}} - \xi_B^k$ which contains J^k. Since ξ_B^k is homotopic to γ_B^k in V_k, from (29) we get

$$\tilde{\Omega} \cap Z = \emptyset.$$

This implies that $\tilde{\Omega}$ contains no critical value of B. It follows that the covering degree of the map

$$B|\eta_B^{k+1} : \eta_B^{k+1} \to \xi_B^k$$

is one. We thus have

$$l_{W_{k+1}}(\eta_B^{k+1}) = l_{V_k}(\xi_B^k). \tag{30}$$

Since $W_{k+1} \subset U_{k+1} = X_B^{k+1}$ we have $\rho_{W_{k+1}} \ge \rho_{U_{k+1}} = \rho_{X_B^{k+1}}$, and thus

$$l_{X_B^{k+1}}(\eta_B^{k+1}) \le l_{W_{k+1}}(\eta_B^{k+1}).$$

This, together with (28) and (30), implies that $l_{X_B^{k+1}}(\eta_B^{k+1}) \le l_{V_k}(\gamma_B^k)$. Since $l_{X_B^{k+1}}(\gamma_B^{k+1}) \le l_{X_B^{k+1}}(\eta_B^{k+1})$, we thus have

$$l_{k+1} = l_{X_B^{k+1}}(\gamma_B^{k+1}) \le l_{X_B^{k+1}}(\eta_B^{k+1}) \le l_{V_k}(\gamma_B^k). \tag{31}$$

By Lemma 4.13, we have

$$\frac{l_{V_k}(\gamma_B^k)}{l_k} = \frac{l_{V_k}(\gamma_B^k)}{l_{U_k}(\gamma_B^k)} \le \prod_{z \in Z} e^{C\mu_z(I^k)}. \tag{32}$$

From (31) and (32) we have

$$\frac{l_{k+1}}{l_k} \le \prod_{z \in Z} e^{C\mu_z(I^k)}.$$

This proves the Claim 2.

From Claims 1 and 2 we have

$$\frac{l_{X_B^N}(\gamma_B^N)}{l_{X_B^0}(\gamma_B^0)} = \left(\prod_{k \in \Lambda_1} \frac{l_{k+1}}{l_k}\right) \cdot \left(\prod_{k \in \Lambda_2} \frac{l_{k+1}}{l_k}\right) \le \big(m(2m-1)\big)^{6(m-1)} \prod_{k \in \Lambda_2} e^{C\mu_z(I^k)}$$

Since

$$\sum_{k \in \Lambda_2} \mu_z(I^k) \le 2\pi,$$

we finally have

$$\frac{l_{X_B^N}(\gamma_B^N)}{l_{X_B^0}(\gamma_B^0)} \le e^{2\pi C}\big(m(2m-1)\big)^{6(m-1)}.$$

This completes the proof of Lemma 4.3. □

4.3. **Proof of Theorem B.** All the arguments used in this section are standard. The readers may find them in several previous literatures, for instance, see [5], [9], and [14].

Let $B \in \mathbf{B}_\theta^m$ be a centered Blaschke product. Recall that $h_B : \mathbb{T} \to \mathbb{T}$ is the circle homeomorphism such that $B|\mathbb{T} = h_B^{-1} \circ R_\theta \circ h_B$ and $h_B(1) = 1$. Now it is sufficient to prove that there exists an $1 < M(m, \theta) < \infty$ depending only on m and θ such that $h_B : \mathbb{T} \to \mathbb{T}$ is an $M(m, \theta)$-quasisymmetric circle homeomorphism. Before that let us introduce some notations and terminologies first.

Let I_1 and I_2 be two arc segments in $\mathbb{T}$. Let $L > 1$. We say I_1 and I_2 are L-comparable if

$$|I_2|/L < |I_1| < L|I_2|.$$

Let $a, b \in \mathbb{T}$ be two distinct points. Recall that we use $[a, b]$ to denote the arc segment in $\mathbb{T}$ which connects a and b anticlockwise and $|[a, b]|$ to denote the Euclidean length of $[a, b]$. For an arc segment $[a, b]$ with $\big|h_B([a, b])\big| \ne \pi$, let us use $\langle a, b\rangle$ to denote $[a, b]$ if $\big|h_B([a, b])\big| < \pi$, and denote $[b, a]$ if $\big|h_B([a, b])\big| > \pi$.

Let $\theta = [a_1, \cdots, a_n, \cdots]$. Let $q_0 = 1$, $q_1 = a_1$, and $q_{n+1} = q_{n-1} + a_{n+1}q_n$ for all $n \ge 1$. For $x > 0$, let $\{x\}$ denote the fraction part of x. For $n \ge 0$, let $\langle q_n\theta\rangle$ denote $\{q_n\theta\}$ if n is even and $1 - \{q_n\theta\}$ if n is odd.

Lemma 4.14. *There exists an $L_0 \ge 2$ independent of θ, such that for all $n \ge L_0$, the following inequality holds,*

$$\langle q_n\theta\rangle < 1/6. \tag{33}$$

Proof. For $n \ge 0$, let p_n/q_n be the nth continued fraction. Let

$$\delta_n = \frac{p_n}{q_n} - \theta.$$

It follows that $|\delta_n| < 1/q_nq_{n+1}$ (for instance, see [12]). This implies that

$$\langle q_n\theta\rangle = |q_n\delta_n| < 1/q_{n+1}.$$

Note that $q_0 = 1$, $q_1 \ge 1$ and $q_{n+2} \ge q_n + q_{n+1}$ for all $n \ge 0$. The lemma then follows by taking $L_0 = 5$. ☐

As an immediate consequence of Lemma 4.14, we have

Corollary 4.2. *Let L_0 be the constant in Lemma 4.14. Then for any $n \ge L_0$ and any $z \in \mathbb{T}$, we have*

$$\langle {R_\theta}^{-q_n}(z), {R_\theta}^{2q_n}(z)\rangle = \langle {R_\theta}^{-q_n}(z), z\rangle \cup \langle z, {R_\theta}^{q_n}(z)\rangle \cup \langle {R_\theta}^{q_n}(z), {R_\theta}^{2q_n}(z)\rangle.$$

Lemma 4.15. *Suppose that $n \ge L_0$. Let $z \in \mathbb{T}$. Then the following two assertions hold.*

1. *Let $I = \langle R_\theta^{-q_n}(z), R_\theta^{2q_n}(z)\rangle$. Then $\{R_\theta^{-k}(I) \mid 0 \le k \le q_{n-2} - 1\}$ is a disjoint family.*
2. *Let $I = \langle z, R_\theta^{q_n}(z)\rangle$. Then $\mathbb{T} \subset \bigcup_{k=0}^{q_n+q_{n+1}-1} R_\theta^{-k}(I)$.*

Proof. The second assertion is standard, for instance, see [5], [9], and [14]. Let us prove the first assertion only. Let us prove it by contradiction. Suppose it were not true. Then there exists a $0 < k < q_{n-2}$ such that

$$R_\theta^{-k}(z) \in \langle R_\theta^{-3q_n}(z), R_\theta^{3q_n}(z)\rangle.$$

It is clear that $R_\theta^{-k}(z) \notin \langle R_\theta^{-q_n}(z), R_\theta^{q_n}(z)\rangle$ by the property of the closest returns. Then we have the following four cases.

In the first case, $R_\theta^{-k}(z) \in \langle R_\theta^{-3q_n}(z), R_\theta^{-2q_n}(z)\rangle$. Let $\xi = R_\theta^{-3q_n}(z)$. Then $R_\theta^{3q_n-k}(\xi) \in \langle \xi, R_\theta^{q_n}(\xi)\rangle$. We then must have $3q_n - k = q_n + q_{n+1}$. Since $q_{n+1} = q_{n-1} + a_{n+1}q_n$, it follows that $a_{n+1} = 1$. So $k = q_n - q_{n-1} \geq q_{n-2}$. This is a contradiction.

In the second case, $R_\theta^{-k}(z) \in \langle R_\theta^{-2q_n}(z), R_\theta^{-q_n}(z)\rangle$. Let $\xi = R_\theta^{-q_n}(z)$. Then $R_\theta^{q_n-k}(\xi) \in \langle R_\theta^{-q_n}(\xi), \xi\rangle$. Since $0 < q_n - k < q_n$, this is impossible.

In the third case, $R_\theta^{-k}(z) \in \langle R_\theta^{q_n}(z), R_\theta^{2q_n}(z)\rangle$. Let $\xi = R_\theta^{-k}(z)$. Then $R_\theta^{2q_n+k}(\xi) \in \langle \xi, R_\theta^{q_n}(\xi)\rangle$. Since $q_n < 2q_n + k = q_n + q_n + k < q_n + q_{n+1}$, this is impossible.

In the last case, $R_\theta^{-k}(z) \in \langle R_\theta^{2q_n}(z), R_\theta^{3q_n}(z)\rangle$. Let $\xi = R_\theta^{-k}(z)$. Then $R_\theta^{3q_n+k}(\xi) \in \langle \xi, R_\theta^{q_n}(\xi)\rangle$. Since $0 < k < q_{n-2}$, we must have $q_n < 3q_n + k < q_n + 2q_{n+1}$. We claim that $3q_n + k = q_n + q_{n+1}$. Let us prove the claim. Assume that the claim were not true. There are two cases. In the first case, we have $3q_n + k = q_n + l$ with $2q_n < l < q_{n+1}$. Then by the property of the closest returns, we have $|\langle R_\theta^{q_n}(\xi), R_\theta^{q_n+l}(\xi)\rangle| = |\langle \xi, R_\theta^{l}(\xi)\rangle| > |\langle \xi, R_\theta^{q_n}(\xi)\rangle|$. This is a contradiction with $R_\theta^{q_n+l}(\xi) = R_\theta^{3q_n+k}(\xi) \in \langle \xi, R_\theta^{q_n}(\xi)\rangle$. In the second case, we have $3q_n + k = q_n + q_{n+1} + l'$ with some $l' > 0$. Since $0 < k < q_{n-2}$, it follows that $l' = 2q_n + k - q_{n+1} < 2q_n + q_{n-2} - q_{n+1} < q_n$. Since both $R_\theta^{q_n+q_{n+1}}(\xi)$ and $R_\theta^{3q_n+k}(\xi)$ belong to $\langle \xi, R_\theta^{q_n}(\xi)\rangle$, it follows that $|\langle \xi, R_\theta^{l'}(\xi)\rangle| = |\langle R_\theta^{q_n+q_{n+1}}(\xi), R_\theta^{3q_n+k}(\xi)\rangle| < |\langle \xi, R_\theta^{q_n}(\xi)\rangle|$. This is again impossible. Thus the claim has been proved and we must have $3q_n + k = q_n + q_{n+1}$.

By the claim we just proved, we have $q_{n+1} = 2q_n + k$. Since $q_{n+1} = q_{n-2} + a_{n+1}q_n$ and $0 < k < q_{n-2}$, we get $a_{n+1} = 1$. This implies that $q_{n-2} = q_n + k$. This is impossible. The proof of the lemma is completed. □

Let $L_0 > 0$ be the universal constant provided in Lemma 4.14.

Lemma 4.16. *There exists a $1 < J(m, \theta) < \infty$ depending only on m and θ such that for every centered Blaschke $B \in \mathbf{B}_\theta^m$, any $n \geq L_0$, and any $z \in \mathbb{T}$, the following two inequalities hold,*

$$1/J(m,\theta) \leq \frac{\left|\langle B^{-q_n}(z), z\rangle\right|}{\left|\langle z, B^{q_n}(z)\rangle\right|} \leq J(m,\theta) \tag{34}$$

and

$$1/J(m,\theta) \leq \frac{\left|\langle B^{q_{n+1}}(z), z\rangle\right|}{\left|\langle z, B^{q_n}(z)\rangle\right|} \leq J(m,\theta). \tag{35}$$

Proof. Let $n \ge L_0$. Take $z_0 \in \mathbb{T}$ such that

$$\big|\langle z_0, B^{q_n}(z_0)\rangle\big| = \min_{z\in\mathbb{T}} \big|\langle z, B^{q_n}(z)\rangle\big|.$$

It follows that

$$C(\langle B^{-q_n}(z_0), B^{2q_n}(z_0)\rangle, \langle z_0, B^{q_n}(z_0)\rangle) < 3. \tag{36}$$

Since θ is of bounded type, there is an integer $0 < \tau(\theta) < \infty$ depending only on θ such that

$$q_n < \tau(\theta) q_{n-2} \tag{37}$$

for all $n \ge 2$. By the first assertion of Lemma 4.15, it follows that for any integer $0 < N \le 5q_n$, the family

$$\{\langle B^{-q_n-k}(z_0), B^{2q_n-k}(z_0)\rangle \,\big|\, 0 \le k \le N\}$$

can be divided into at most $5\tau(\theta)$ disjoint sub-families. By (36) and by applying Lemma 4.2 successively at most $5\tau(\theta)$ times, we get a constant $0 < P_1(m,\theta) < \infty$ depending only on m and θ such that the following inequality

$$C(\langle B^{-(l+1)q_n}(z_0), B^{(2-l)q_n}(z_0)\rangle, \langle B^{-lq_n}(z_0), B^{(1-l)q_n}(z_0)\rangle) < P_1(m,\theta) \tag{38}$$

holds for $0 \le l \le 5$.

We claim that there exists a $0 < P_2(m,\theta) < \infty$ depending only on m and θ such that any two of the following six arc segments

$$\langle B^{-lq_n}(z_0), B^{(1-l)q_n}(z_0)\rangle,\ 0 \le l \le 5, \tag{39}$$

are $P_2(m,\theta)$-comparable. Let us prove the claim. It suffices to prove that among these six arc segments, any two adjacent ones are $P_1(m,\theta)$-comparable. Let us prove this only for the pair of adjacent arc segments

$$\langle z_0, B^{q_n}(z_0)\rangle \text{ and } \langle B^{-q_n}(z_0), z_0\rangle.$$

The same way can be used for the other four pairs of adjacent arc segments. By taking $l = 0$ in (38) we get

$$C(\langle B^{-q_n}(z_0), B^{2q_n}(z_0)\rangle, \langle z_0, B^{q_n}(z_0)\rangle) < P_1(m,\theta).$$

This implies that

$$\frac{|\langle z_0, B^{q_n}(z_0)\rangle|}{|\langle B^{-q_n}(z_0), z_0\rangle|} < P_1(m,\theta). \tag{40}$$

By taking $l = 1$ in (38) we get

$$C(\langle B^{-2q_n}(z_0), B^{q_n}(z_0)\rangle, \langle B^{-q_n}(z_0), z_0\rangle) < P_1(m,\theta).$$

This implies that

$$\frac{|\langle B^{-q_n}(z_0), z_0\rangle|}{|\langle z_0, B^{q_n}(z_0)\rangle|} < P_1(m,\theta). \tag{41}$$

From (40) and (41) it follows that the two adjacent arc segments $\langle z_0, B^{q_n}(z_0)\rangle$ and $\langle B^{-q_n}(z_0), z_0\rangle$ are $P_1(m,\theta)$-comparable. The same way can be used to prove the other four adjacent arc segments are also $P_1(m,\theta)$-comparable. The claim then follows by taking $P_2(m,\theta) = P_1^5(m,\theta)$.

Let

$$l_0 = \big|\langle z_0, B^{q_n}(z_0)\rangle\big|.$$

By the choice of z_0, it follows that l_0 is the minimum of the length of the six intervals in (39). By the Claim we proved above, we have

$$P_2(m,\theta)^{-1} \cdot l_0 \le |\langle B^{-lq_n}(z_0), B^{(1-l)q_n}(z_0)\rangle| \le P_2(m,\theta) \cdot l_0,\ 0 \le l \le 5. \tag{42}$$

For any $z \in \mathbb{T}$, it follows from the second assertion of Lemma 4.15 that there is an $0 \le i < q_n + q_{n+1}$ such that $B^i(z) \in \langle B^{-5q_n}(z_0), B^{-4q_n}(z_0)\rangle$. We then have the following two cases.

In the first case, there is some $1 \le j \le 3$ such that

$$\big|\langle B^{i+jq_n}(z), B^{i+(j+1)q_n}(z)\rangle\big| < l_0/2.$$

This implies

$$C(\langle B^{i+(j-1)q_n}(z), B^{i+(j+2)q_n}(z)\rangle, \langle B^{i+jq_n}(z), B^{i+(j+1)q_n}(z)\rangle) < 3. \tag{43}$$

Since $0 \le i < q_n + q_{n+1}$ and $1 \le j \le 3$, by (37) we have

$$0 < i + jq_n < 4q_n + q_{n+1} < (4\tau(\theta) + \tau(\theta)^2)q_{n-2}.$$

By (43) and the first assertion of Lemma 4.15, and by applying Lemma 4.2 successively at most $(4\tau(\theta) + \tau(\theta)^2)$ times, we get a constant $P_3(m,\theta) > 0$ depending only on m and θ such that

$$C(\langle B^{-q_n}(z), B^{2q_n}(z)\rangle, \langle z, B^{q_n}(z)\rangle) < P_3(m,\theta). \tag{44}$$

In the second case, we have

$$\big|\langle B^{i+jq_n}(z), B^{i+(j+1)q_n}(z)\rangle\big| \ge l_0/2$$

for each $j = 1, 2, 3$. This, together with (42), implies that there exists a $0 < P_4(m,\theta) < \infty$ depending only on m and θ such that

$$C(\langle B^{i+q_n}(z), B^{i+4q_n}(z)\rangle, \langle B^{i+2q_n}(z), B^{i+3q_n}(z)\rangle) < P_4(m,\theta). \tag{45}$$

Since $0 < i + 2q_n < 3q_n + q_{n+1} < (3\tau(\theta) + \tau(\theta)^2)q_{n-2}$, By (45) and the first assertion of Lemma 4.15, and by applying Lemma 4.2 successively at most $(3\tau(\theta) + \tau(\theta)^2)$ times, we get a constant $0 < P_5(m,\theta) < \infty$ depending only on m and θ such that

$$C(\langle B^{-q_n}(z), B^{2q_n}(z)\rangle, \langle z, B^{q_n}(z)\rangle) < P_5(m,\theta). \tag{46}$$

Let $P_6(m,\theta) = \max\{P_3(m,\theta), P_5(m,\theta)\}$. From (44) and (46) it follows that in both the cases, the following inequality holds,

$$C(\langle B^{-q_n}(z), B^{2q_n}(z)\rangle, \langle z, B^{q_n}(z)\rangle) < P_6(m,\theta). \tag{47}$$

Since (47) holds for an arbitrary $z \in \mathbb{T}$, by considering the point $B^{-q_n}(z)$, we get

$$C(\langle B^{-2q_n}(z), B^{q_n}(z)\rangle, \langle B^{-q_n}(z), z\rangle) < P_6(m,\theta). \tag{48}$$

From (47) we have $|\langle z, B^{q_n}(z)\rangle| < P_6(m,\theta)|\langle B^{-q_n}(z), z\rangle|$. From (48) we have $|\langle B^{-q_n}(z), z\rangle| < P_6(m,\theta)|\langle z, B^{q_n}(z)\rangle|$. This implies that for any $z \in \mathbb{T}$, the inequality

$$1/P_6(m,\theta) \le \frac{\big|\langle B^{-q_n}(z), z\rangle\big|}{\big|\langle z, B^{q_n}(z)\rangle\big|} \le P_6(m,\theta) \tag{49}$$

holds for all $n \ge L_0$. This proves the first assertion of Lemma 4.16 by taking $J(m,\theta) = P_6(m,\theta)$.

Now let us prove the second assertion of Lemma 4.16. Note that

$$\langle z, B^{-q_{n+1}}(z)\rangle \subset \langle z, B^{q_n}(z)\rangle,$$

so from (34), we have

$$\big|\langle B^{q_{n+1}}(z), z\rangle\big| \le J(m,\theta)\big|\langle z, B^{-q_{n+1}}(z)\rangle\big| < J(m,\theta)\big|\langle z, B^{q_n}(z)\rangle\big|,$$

and this implies the right hand of (35). To prove the left hand, Note that

$$\langle z, B^{q_n}(z)\rangle \subset \bigcup_{0\le i\le b(\theta)} \langle B^{-iq_{n+1}}(z), B^{-(i+1)q_{n+1}}(z)\rangle,$$

where $b(\theta) = \sup\{a_n\}$. This implies that

$$\big|\langle z, B^{q_n}(z)\rangle\big| \le \sum_{0\le i\le b(\theta)} \big|\langle B^{-iq_{n+1}}(z), B^{-(i+1)q_{n+1}}(z)\rangle\big|.$$

For $0 \le i \le b(\theta)$, by applying (34), we have

$$\big|\langle B^{-iq_{n+1}}(z), B^{-(i+1)q_{n+1}}(z)\rangle\big| \le J(m,\theta)^{i+1}\big|\langle B^{q_{n+1}}(z), z\rangle\big|.$$

Therefore, we get

$$\frac{\big|\langle z, B^{q_n}(z)\rangle\big|}{\big|\langle B^{q_{n+1}}(z), z\rangle\big|} \le \sum_{0\le i\le b(\theta)} J(m,\theta)^{i+1}.$$

This proves the second assertion of the Lemma by modifying $J(m,\theta)$. This completes the proof of Lemma 4.16. □

Now let us prove Theorem B. Let $L_0 > 0$ be the integer in Lemma 4.14. Take an arbitrary $z \in \mathbb{T}$ and an arbitrary $0 < \delta < 2\pi$.

First let us assume that one of $\langle z, B^{q_{L_0}}(z)\rangle$ and $\langle z, B^{q_{L_0+1}}(z)\rangle$ is contained either in $[e^{-i\delta}z, z]$ or in $[z, e^{i\delta}z]$. With this assumption let us show that there exists an $1 < M_1(m,\theta)$ depending on only on m and θ such that

$$M_1(m,\theta)^{-1} < \frac{\big|h_B([z, e^{i\delta}z])\big|}{\big|h_B([e^{-i\delta}z, z])\big|} < M_1(m,\theta). \tag{50}$$

Without loss of generality, let us assume that

$$\langle z, B^{q_{L_0}}(z)\rangle = [z, B^{q_{L_0}}(z)] \subset [z, e^{i\delta}z]. \tag{51}$$

Since θ is of bounded type, by Lemma 4.16, there is an integer $N_1(m,\theta)$ depending only on m and θ such that

$$\big|\langle B^{q_{L_0+1+2N_1(m,\theta)}}(z), z\rangle\big| \le \big|\langle z, B^{q_{L_0}}(z)\rangle\big|.$$

We thus have

$$\langle B^{q_{L_0+1+2N_1(m,\theta)}}(z), z\rangle \subset [e^{-i\delta}z, z]. \tag{52}$$

From (51) we have

$$\langle q_{L_0}\theta\rangle \cdot 2\pi \le h_B([z, e^{i\delta}z]) < 2\pi. \tag{53}$$

From (52), we have

$$\langle q_{L_0+1+2N_1(m,\theta)}\theta\rangle \cdot 2\pi < h_B([e^{-i\delta}z, z]) < 2\pi. \tag{54}$$

We thus have (50) in this case by taking

$$M_1(m,\theta) = \min\{\frac{1}{\langle q_{L_0}\theta\rangle}, \frac{1}{\langle q_{L_0+1+2N_1(m,\theta)}\theta\rangle}\}.$$

Now assume that neither of $\langle z, B^{q_{L_0}}(z)\rangle$ and $\langle z, B^{q_{L_0+1}}(z)\rangle$ is contained in $[e^{-i\delta}z, z]$ or $[z, e^{i\delta}z]$. Let $k \ge L_0 + 2$ be the least integer such that either $[e^{-i\delta}z, z]$ or $[z, e^{i\delta}z]$ contains $\langle z, B^{q_k}(z)\rangle$. Suppose that

$$\langle z, B^{q_k}(z)\rangle = [z, B^{q_k}(z)] \subset [z, e^{i\delta}z]. \tag{55}$$

The other cases can be treated in the same way. Then by the assumption and the definition of k, we have

$$[z, B^{q_k}(z)] \subset [z, e^{i\delta}z] \subset [z, B^{q_{k-2}}(z)] \tag{56}$$

and

$$[e^{-i\delta}z, z] \subset [B^{q_{k-1}}(z), z]. \tag{57}$$

Let $J(m,\theta)$ be the constant in Lemma 4.16. By (56) and Lemma 4.16, it follows that

$$\big|[B^{q_{k-1}}(z), z]\big| \le J(m,\theta)\big|[z, B^{q_k}(z)]\big| \le J(m,\theta)\delta. \tag{58}$$

Note that for $n \ge L_0$,

$$\langle B^{q_{n+2}-q_{n+1}}(z), B^{q_{n+2}}(z)\rangle \cup \langle B^{q_{n+2}}(z), z\rangle \subset \langle B^{q_n}(z), z\rangle. \tag{59}$$

By the first assertion of Lemma 4.16 we have

$$|\langle B^{q_{n+2}-q_{n+1}}(z), B^{q_{n+2}}(z)\rangle| \ge J(m,\theta)^{-1}|\langle B^{q_{n+2}}(z), B^{q_{n+2}+q_{n+1}}(z)\rangle|. \tag{60}$$

and

$$|\langle B^{2q_{n+2}}(z), B^{q_{n+2}}(z)\rangle| \ge J(m,\theta)^{-1}|\langle B^{q_{n+2}}(z), z\rangle|. \tag{61}$$

By the second assertion of Lemma 4.16, we have

$$|\langle B^{q_{n+2}}(z), B^{q_{n+2}+q_{n+1}}(z)\rangle| \geq J(m,\theta)^{-1}|\langle B^{2q_{n+2}}(z), B^{q_{n+2}}(z)\rangle|. \tag{62}$$

From (60)-(62), we have

$$|\langle B^{q_{n+2}-q_{n+1}}(z), B^{q_{n+2}}(z)\rangle| > J(m,\theta)^{-3}|\langle B^{q_{n+2}}(z), z\rangle|. \tag{63}$$

From (59) and (63) we have

$$\big|\langle B^{q_n}(z), z\rangle\big| \geq (1+J(m,\theta)^{-3})\big|\langle B^{q_{n+2}}(z), z\rangle\big| \tag{64}$$

holds for all $n \geq L_0$. Let $N_2(m,\theta) > 0$ be the least positive integer such that

$$\big(1+J(m,\theta)^{-3}\big)^{N_2(m,\theta)} > J(m,\theta).$$

From (58) and (64), it follows that

$$[B^{q_{k-1+2N_2(m,\theta)}}(z), z] \subset [e^{-i\delta}, z]. \tag{65}$$

From (56) we have

$$\langle q_k\theta\rangle \cdot 2\pi \leq h_B([z, e^{i\delta}z]) \leq \langle q_{k-2}\theta\rangle \cdot 2\pi. \tag{66}$$

From (57) and (65), we have

$$\langle q_{k-1+2N_2(m,\theta)}\theta\rangle \cdot 2\pi < h_B([e^{-i\delta}z, z]) < \langle q_{k-1}\theta\rangle \cdot 2\pi. \tag{67}$$

Since θ is of bounded type, from (66) and (67), it follows that there exists an $1 < M_2(m,\theta) < \infty$ depending only on m and θ such that in this case

$$M_2(m,\theta)^{-1} < \frac{\big|h_B([z, e^{i\delta}z])\big|}{\big|h_B([e^{-i\delta}z, z])\big|} < M_2(m,\theta).$$

Theorem B then follows by taking $M(m,\theta) = \max\{M_1(m,\theta), M_2(m,\theta)\}$.

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办公用房

张高飞

2013 年 1 月 18 日

小时候家里经济条件不好。那时每人的口粮都有定额。由于我的两个哥哥比我大好多，而且成天帮父母做很重的活儿，每当过年过节有些好吃的东西时，父母会稍稍分给两个哥哥多一些。那时一遇到这样的场面我就很不开心。而事情的结局往往是父母把最多的那份又分给了我。可记忆中我从来没有吃掉最多的这一份，我最后还是把它给了我年龄最大的哥哥，毕竟他看上去已是个大人了。我想当时自己想要的只是想知道父母并没有从内心里偏向谁。

今天分办公室的这一幕竟让我又回想起这些已尘封多年的往事。因为自己执拗的表现，书记和主任把我的名字调到了很靠前的位置。看到那张新的名单我意识到这并不是我真正想要的。我现在的办公室已足够好了。我从未打算搬到蒙明（民）伟楼去。最主要的是我从未在办公室里做过哪怕一件有意义的事情。相比我，很多老师更需要学习，因此更需要一间条件好一些的办公室。我想要的只是确认把我排在教授中的最后一名并不是系里在有意偏向谁。我已经知道结果了，我很满意。因此明天监考结束我就回家了。我本来就没打算去点房子。我还用现在的这间办公室。

希望大家到时都能分到自己想要的房间，并提前祝大家春节愉快。

给梅加强老师的一封信

张高飞

加强：您好！

本周三在班车上我们聊的很愉快。其间您告诉我去年尤老师被评为最受学生欢迎的老师。您说对于这件事情我们两人要好好反思一下：我们都是给大一上数学分析课的，可为什么我们两人都没有获此殊荣。回来之后我仔细的思考了这一问题。现把我自己的想法总结如下。仅供您参考。

就对教材的熟练程度上，以及解题能力上，我想尤老师都无法与你我相比。尤老师自己也说，他在大学时成绩很差，只是到了大四成绩才一下子好了起来。可我们都知道，本科大学的第四年除了一些音乐，美术之类的基本上就没什么主课了。而数学分析课是大一大二开的。可想在三十年以前尤老师就没有学好这门课程。而你我则不同。如同我们的名字一样，你我都属于那种志向高远，精益求精的人。讲数学分析这种基础课程，我俩自然都是驾轻就熟，游刃有余。说到这里，您自然会问，既然如此，那为何尤老师讲的课最受学生的欢迎呢？

这其实正是问题的关键所在。尤老师深知自己对这门课程掌握的不够火候，因此他讲课时只讲那些自己有相当把握的。比如凡是证明过程超过十行的，尤老师都留给习题课上讲。凡是计算中间需要技巧的，尤老师就循循善诱的告诉大家，他在课上主要讲思想，这些技巧上的东西就留给习题课上讲吧。如此一来，尤老师的课不仅讲的轻松，学生听的也轻松。就连工科毕业的辅导员杨靖在听完尤老师的课后，都感慨的说：原来数学系的数分课比她们当初学的大学数学容易多了。当然尤老师有时也会小试牛刀一把。比如有的同学会在课间休息时请教课后的习题。在确信自己有十足的把握拿下该题时，尤老师总会毅然走上讲台，奋笔疾书。每当这种时刻，重修的同学都会背起书包，悄悄离开了教室。因为他们知道在接下来的一节课，尤老师会一直专注在这道题上。直到下课铃声响起。然而让尤老师在教学上取得极大成功的却正是他的这种教学风格。几乎所有同学都觉得尤老师讲课思路清晰，浅显易懂。就连那些入学时被调剂到数学系的学生都对数学产生了极

大的兴趣。特别是当他们目睹了头顶无数光环的尤老师在一个并不很难的问题上久攻不克时，更是信心倍增。

尤老师的课与其说是一堂数学课，倒不如说是一堂励志课。大家都凝视着尤老师，对自己的未来充满了梦想。然而在大一的第二学期所有这一切都变了。你教一班，我教二班。你我讲课行云流水，面面俱到。大多数同学都不能当堂消化所讲内容。而你不时在黑板的角落处写下的思考题，好多人学期结束时也没做出来。上学期大家都觉得自己是天才，这学期似乎还不如普通人。前后对比，尤老师与我俩形成了鲜明的对比。在埋怨你我的同时，大家更加想念那个能让他们自信满满的尤老师。而恰好此时，学校开展了评比活动。结果可想而知。

据杨靖告知，在抽查的 120 人中，有 116 人把票投给了尤老师。有 3 个人投票给我。他们都是我河北老乡。只有一人投了你一票。但那个人不是你的湖北老乡。因为你从来没告诉大家你是湖北人。那个人是辅导员杨靖。她知道你一向工作很认真，怕你一下子接受不了。好了，加强，就说这么多吧。我们都还年轻。只要我们好好努力，未来还有好多机会。希望就在前方。

此致

敬礼！

高飞

给孙老师的一封信

张高飞

孙老师：

我一直发自内心的敬仰您。为此我还买了好几本组合数学的书。可我属于无志者常立志的那种人。我一有时间就去打球了。因此没有深入的读任何一本书。在我的“研究生选专业指南”一帖中我把您的专业比作中学数学完全是为了娱乐大家。同样把何老师的计算数学排在最后也是觉得好玩。因为我想到您和何老师都智力超常，成就斐然，因此大家一看就觉得这是在开玩笑。而您们也应不会在意的。

我看到您说自己很苦恼的那一个帖子。我有个建议，不知是不是合适。我觉得要是我的话，我就写一篇论文，讲明白这个如何可用来获得该类 π 级数的方法。任何人读了您的论文后，都能向您一样提出这样的 π 级数。这不就是我们对数学的发展想要做的贡献吗？这样一来，您也可以腾出更多的时间来做其它的研究，不会过一些日子就用同样的办法给出一个新的 π 级数，而且还说是靠灵感得来的，让我们其他老师显得就像白痴一样。您说呢？

我年轻时象您现在一样，喜欢在众人面前表现自己是多么多么的超人一等。可当我发觉自己本来很茂密的头发随着岁月的流逝一把一把的脱落时，我的内心逐渐成熟起来。我变得低调，淡定，谦虚而且内敛。这就是现在的我。我觉得孙老师您虽在学术上堪为全系人的典范，可在为人处事上应向我学习啊。

此致

敬礼！

高飞

给尤老师的一封信

张高飞

2010 年 3 月 17 日

尤老师，

您好！

好多学生说我给他们带来了快乐。可快乐的根源却是您。每一次把您顶上十大，您都不介意。反而兴高采烈的到系里来喊我去打球。那一刻我才意识到您心里原来很想上十大。可即使真的如此，您也应该矜持一些才对。再说我也需要时间寻找素材啊。现在的小孩很聪明。和您们那时候不一样了。再说他们对我写的帖子期望很高。就是忽悠也要本着对他们负责的态度啊。

说起您的球技，我觉得您真的提高很多。我知道您一直试图忽略我们之间的差距。我经常同老纪到您那儿打球。可我和您们能在一起打球，并不代表我们的水平就在一条线上。就像胡主席春节期间到一些乡镇去慰问，和当地的干部们在一起吃顿饭一样。您和老纪的实力的确在伯仲之间。可我却听到了您小心翼翼的说我们三人的水平是个小等差数列。等差数列也就算了。您真的不该强调那个“小”字。您记得那天我们最后的那场比赛吗？您本来以 10：1 领先。可最后却以 10：12 输掉了比赛。我就想暗示您如果我想赢您的话，只要一个条件就够了。那就是比赛还没结束。

再说点认真的。我看到您和孙老师要到仙林做个报告。我想您俩的报告一定很精彩。我们好多年轻老师也想去。可我们希望不要在这种报告的报酬上对正教授和副教授有所区分。我同一些老师说过这个话题。我们都有个共同的感受：因为那样的话就显得我们这些副教授的水平不如那些正教授。可您知道，事实并非如此。就好像我们明明是优等生。可老师偏偏说我们不如那些差等生。您说我们心里会好受吗？当然我们副教授中也不乏一些没骨气的人。为了 100 块钱积极响应系里的号召。

上次我们系全国排名第 12 名。有许多学生在小百合上表示很伤心。可是那些正教授中没有一个站出来安慰一下这些学生，承担这个责任。若把正教授比做正规军，我们这些副教授就好比民兵。您们论装备比我们好，论战

斗力，却还不如我们。尤老师，就算我谦虚，其他人也不服气啊。

好了，太晚了。先说这么多了。

此致

敬礼！

高飞

工作量的计算

张高飞

2011 年 3 月 5 日

去年一年我只在遍历论上发表了一篇科研论文，而且还是第二作者。因此科研工作量是零。年底时我看到了贴在墙上的工作量统计表。我差不多是全系最低的。心里很不是滋味。

回想起自己在过去的一年里其实也做了很多工作。特别是上半年我一连发了很多大家都喜闻乐见的帖子。其中“给梅加强老师的一封信”和“研究生选专业指南”可谓是这些帖子中的代表作。这些帖子通过互联网传遍大江南北。许多读者都向我表示这些帖子让他们受益良多。我暑期在外地参加活动时，有人把这些帖子打印出来。好几次上面的人作报告，下面的学生在聚精会神的传阅着我的帖子。报告结束时，他们没有与作报告的人做任何交流，却纷纷找我在帖子上签字。并表示通过读我的帖子，他们对人生有了新的感悟。他们都一致表示希望我在学习工作之余能多写一些这样的帖子。

我因此特别恳请系里在年底统计工作量时考虑一下我的这些帖子。小百合虽不是 SCI 杂志，可我想与其在那些严肃的学术刊物上灌水，倒不如把水灌在小百合上。毕竟百合是我们南大的百合，我们灌自己的水，别人也说不出什么。所以我提议系里可否有这样的政策：

凡是被顶上十大之首的贴子按一篇三类的 SCI 论文计。

我静听系里的答复。

纪念那些过去的日子

张高飞

来南大已八个年头了。那时自己算是系里的年轻老师。整天和郭学军老纪呆在一起。当时老纪的爱人还没调过来。而郭学军那时不知为什么过着好像是单身的生活。我爱人一直在外地。于是我们三人不谋而合的凑到了一起。三个人经常为一些并不可笑的笑话笑得前仰后合，往往笑过之后讲笑话的那个说笑话还没开始，刚讲的只是个铺垫，于是另外两个耐心的等他讲完。接着再重新笑一遍。

那时的郭学军并不像现在这么斯文。尽管他早就开始写诗并在公共场合下背诵诗歌。但我一直认为他这样做只是为了吸引中文系那几个女老师的注意。更多的时候他会在班车上大声的和你讲什么时候容易受孕，什么时候精子的成活率较高。每当这样的场合我都尽量做出一副严肃的表情，不时的点头配合，努力为这次谈话营造一些学术的气氛。

记得一次刚一坐下来，他突然问了一句：高飞，你喜欢男人唱的歌吗？这句话让前后左右的几个人都朝我们这儿看过来。其实完全可以换个问法。比如说你最喜欢的男歌手是谁？可那时的郭学军就是这样。一句话让人瞬时间不知如何应对。相比之下，老纪总是稳稳当当。他自己很少主动创作什么笑话。他总是在别人讲了笑话后很实在的捧场，同时也积极地为别人在笑话的创作上提供素材。就这样在不知不觉中过了好多年。

直到某一天老纪告诉我他爱人调到南京工作了。又过了不久得知郭学军重新组织了家庭。只有我没有变化。仍然是两地生活。忘了从什么时候起我们不再像从前那样经常在一起说些无厘头的笑话。有时见了面也是说不了几句话就匆匆散了。

所有这一切都是在不声不响中发生的。似乎本应该就是这样。尽管这时又有些年轻老师进来了。如苗栋和石亚龙可和他们在一起时感受不到以往的那种快乐。孙永忠和尤老师偶尔也加入我们。但他们两人总是一副知识分子的模样。他们那些貌似很有品味的谈吐让我更加怀念以前那些毫无意义的无所顾忌的笑话。当意识到那些曾今不为别人注意的却让自己内心产生

极大震撼的场景今生再不会出现时，我的情绪在刹那间低落到了极点。

教学论文

张高飞

2011 年 10 月 10 日

刚到南大时我有点不适应。觉得当了老师还是被人管。当时系里的教务主任是丁南庆老师。短短的一个学期竟然和我谈了三次话。具体内容都已忘了。就记得最后一次谈话我实在气愤不过把书狠狠摔在桌子上。

为此在那些日子里我编写了个小册子，零零种种一共收集了七十多个教学上可能出现的问题。对每个问题我都精心的准备了解答。比如说由于记错课表而导致旷课该怎么办？提前下课被人投诉到教务处该怎么办？自主调课让教务处查到怎么办？等等。每一条解答我都精心设计，再三推敲。为的是确保万一用到时必能做到以理服人，以情感人，让对方觉得于情于理都无懈可击。然而这本花了我一年半心血的小册子自从编好之后再也没有人找我谈话了。看着这本已泛黄的小册子，心里突然有种英雄无用武之地的失落感。

前些天我整理书桌又看到了这个小册子。不知不觉得翻到了最后一页。那是论述上课时少讲些习题而多讲些笑话的合理性。大致意思如下。一个班上的学生可分为三类。第一类的学生不用老师讲就会。第二类的学生老师讲了也不会。第三类的学生老师讲了才会。第一类学生本来就会。再讲那些习题会让他们产生审美疲劳。不如讲些笑话放松一下。第二类学生既然讲了都不会，再讲无疑是对他们心里和生理的双重折磨。从人道主义出发也应该讲些笑话。最微妙的是第三类。这部分学生将来是不能从事数学研究的。他们将来是社会的中坚力量，比如工程师，企业家，行政官员等等。可是如果这时耐心的把题目讲了，并且还讲得让他们听明白了，这就会使得这部分同学误以为他们在数学上很有天分。而这种错觉会让他们在大学毕业后选择一条不适合自己发展的人生道路。从某种意义上讲这样做是害了他们。相反，若只讲些笑话，这部分学生由于一直没搞明白，时间一长，对数学便失去了兴趣。于是他们毕业后有的去学金融，有的去学管理，有的直接就考公务员了。由此可见讲笑话无形中帮助了他们正确的确定了自己的人生道路。. . .

吃饭时我讲给几个同事听。他们都不置可否。好像他们从来没思考过类似的问题。昨天我爱人说她们学校评职称需要发表教学论文，让我帮着写一篇。我把上面的内容整理了一下。题目是“教学模式改革之探索系列-1”。今天早上我到建行交了800元版面费。下午就收到编辑的接收函。另附审稿人评语为：该文思想相当前卫，读后让人耳目一新。推荐在贵刊发表。

如何选择数学方向

张高飞

正如大家所知，代数几何是现代数学的主流。当代大多数一流的数学家都工作在这一领域。因此如果你觉得自己天赋异禀，并在代数，几何与分析各方面都有着扎实的基础，我建议你绝不要浪费自己的天赋：应义无反顾的选择代数几何这一专业。当然把代数，几何与分析这三门基础功课同时学好的人很少。比如有些同学有着很好的分析功底，但代数中的抽象思维能力却相对显得薄弱。如果是这样的话，我建议你选择分析方面的专业，比如：复分析，分形，调和分析或微分方程。

如果你代数和分析都不怎么样，可却在几何方面有着良好的感觉，要是这样的话，我建议你应和梅加强老师好好探讨一下。让他帮你判断一下看自己是不是可以学习几何。

除以上三部分同学之外，还有这样的一部分同学：他们对代数，分析与几何都不擅长，但却一直坚信自己在数学上仍能有所作为，并幻想有朝一日成为中国数学界的中流砥柱。如果你属于这部分同学中的一位的话，我建议你选择动力系统。动力系统这一学科其实就是专门为这部分同学开设的。

当然即使是动力系统也不是人人都能学的。因为动力系统需要大量的微积分。可总有那么一部分同学还没来得及把极限的概念搞清楚就大学毕业了。如果你不巧就是这样的一位同学，也就是说你大学四年压根儿就没学数学，但仍希望自己将来能在数学上一展宏图的话，我建议你选择组合数学这一专业。这一专业的特点就是它只用到中学的数学。如果你在中学时参加过数学竞赛并获过奖项的话，这一学科正是你大展身手的地方。

我想大多数同学看到这儿之前已经找到了适合自己的专业了。可若仍有人羞怯的说他在中学时早恋，因此连中学的数学也没学好，我想告诉这部分同学不要怕。在我们系有专门为你们开设的一个专业：统计学。这一学科只要求懂得小学数学中的加减乘除四则运算就够了。更重要的是，选择这一专业的大多都是女同学。在你准确无误的把成千上万个数据加起来并娴熟的计算出他们的均值时，你也赢得了众多师姐师妹的芳心：短短三年的研究

生生活或许能让你再次体会一次那如花美眷，似水流年的往事。。。

最后这一条是专门针对那些悲情人物的。他们连小学的数学也没学好。不要说把上千个数加起来，就是把两个数加起来，对他们来说都是件很吃力的事。然而这一切丝毫没有削弱他们对数学的一片痴情。他们日日夜夜泡在图书馆里。他们翻阅了所有的数学文献，却从未找到一本能读懂的。但他们仍坚持不懈，为的就是找到一个适合自己的专业。他们的行为感动了上帝。上世纪的某一天，上帝为他们创造了一台机器帮他们计算。这就是计算机。借助计算机，他们可以很快的进行加减乘除的运算。这就是计算数学。

为 05 级同学送行

张高飞

2009 年 7 月 1 日

前几天大礼堂有一场毕业晚会，胡泽春拉我同去。到了才知道还要票。门口有四张桌子排成两排，四个学生坐在后面负责查票。许多老师和家属都被挡在外面。我俩只有一张票。我就让胡泽春进去了。一出礼堂我觉得不甘心，我想一定要找个办法进去。

我从礼堂门口外面快步走到第一张桌子旁边，一边敲着桌子一边急促的说，“哎，同学，快点，把桌子搬到舞台上去，还缺个道具”。因为晚会马上要开始了，那个同学立刻站起来抱着桌子就走。其他几个同学纷纷表示愿意帮忙，我示意他们一个人就够了。

学生到底年轻，虽抱着一张桌子，还能健步如飞，三步两步就把我甩在了后面。这时所有的灯光都聚集在舞台中央，台下黑乎乎一片。我见有一个地方有稍许亮光。我走过去坐了下来。果然旁边坐的就是胡泽春。

这时那个学生已走上了舞台。他抱着桌子，环顾四周，不知放在什么地方合适。旁边的主持人惊讶的看着他问：“同学，这是怎么回事?”

“不是要做道具吗?”

“谁说要道具了，独唱还要什么道具呀，赶快搬走，马上要开始了!?”

那个同学无辜的朝台下望了望，一脸困惑。抱起桌子沿原路返回去了。那一刻我心里突然有一丝内疚。有一种冲动让我站起来和他一起把桌子搬出去，可我最终都没站起来，一直到晚会结束。

对那晚的演出我已毫无印象了。只记得胡泽春一直在旁边动情的拍着手，呵呵的笑着。看着他阳光明媚的笑脸，明白一个道理：**人要快乐，就必须简单一些。**

尤老师的乒乓球之路

张高飞

我们系里和所里各有一张乒乓球台子。水平低一些的就在系里玩，高一些的就在所里。就像CBA与NBA，分得很清楚。比如，何老师和孙老师就总在系里。汪老师，钟老师和武海军他们一般都在所里。在系里打得好的，比如老纪，有时就去一下所里。而状态不能保持的 ，比如朱老师，现在就只能在系里打一打。 当然也有例外，比如尤老师一直在所里打。那只是因为他的办公室在所里。

尤老师不仅自己对乒乓球保持着浓厚的兴趣，而且也热心关注我们系的乒乓球事业。比如在最近的一场师生对抗赛中，尤老师带病参加了比赛。虽然以零比三惜败给我系一个女同学，可是他那种战斗到底永不言输的精神给我们留下了深刻的印象。

尤老师除了训练的很刻苦之外，还请人买了很贵的球拍，同时从网上下载了教学录像来纠正自己的动作。尤老师的心中有一个梦想：就是希望有一天能够得到大家的认可，从而名正言顺的加入到我们系的高手俱乐部。因为尤老师是主任，所以汪老师，朱老师他们和尤老师比赛时总打假球。有时故意回一个很高的球，等尤老师把球扣过来时再大声喊“好球”！有一次刚把球回过去，就开始喊“好球”。结果尤老师一拍子把球抡在了网子上。我在旁边实在看不下去了。就说：“尤老师，他们是说这颗球是刚买的，是个好球”。

尽管如此，在过去的一年里有尤老师的乒乓球技术还是取得了可喜的进步。比如他学会了拉下旋球，学会了搓球，并且不再吃我发的不转球。有时我甚至想或许真的有那么一天， 即使我不让球也输给了尤老师。 那该是怎样的一个世界啊。

动力系统讨论班

张高飞

那还是在浦口。我们当时和大一学生有个一维动力系统的讨论班。尽管预先借好了教室，可有一次还是遇到了类似的情况：教室里坐满了人，讲台上的老师手舞足蹈的正讲着起劲，完全没有短时间内停下来的意思。

经过了片刻的思索，我在字条上写下了几个教室的房间号，然后很从容的走了进去。我把条子递给那位老师，并以教务处工作人员的口气告诉他们这间教室的天花板在装修时没有加固好，装修人员即时就到。我特别叮嘱大家走时动作要轻些。因为脚步声太重会导致天花板脱落下来。接下来我指挥着人们有秩序的离开。每个人都把书包顶在头上，完全没有注意到这间教室并没有什么天花板。事隔多年，我还记得大家出门时对我流露出的那种感激的眼神。

几分钟之后我们讨论班上的学生就到齐了。他们完全不知道刚才发生过的事情